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Determinations Based on Duplication of Readings

Janace A. Speckman

(February 25, 1964)

This paper is concerned with a statistical estimation procedure in which measurements of a quantity are taken until two identical readings are obtained; this duplicated value is then taken as the estimate of the magnitude of the quantity concerned. The properties of this estimation procedure have been investigated numerically, under the assumptions that the individual observations are rounded values of measurements which have a normal distribution, and this estimator is compared with the arithmetic mean of two observations. It is shown that an arithmetic mean of two observations from the rounded distribution is almost always superior to the estimator described above. The exception is where the rounding interval is so wide and the rounding lattice is so advantageously placed that the only real reason for taking repeat measurements would be as a protection against gross errors.

The *1961 Book of ASTM Standards* [3, p. 1131] gives tentative methods for calibrating upright tanks. These include a proposed method for obtaining circumferential measurements on certain types of tanks which consists, briefly, of wrapping the measuring tape around the tank (at some specified position) with tension applied to the tape, taking a reading to the nearest 0.005 foot, releasing then reapplying the tension, taking another reading, etc., until two identical readings are obtained. The value of the equal readings is then recorded as the circumferential measurement.

The method of estimation used above is familiar to all of us, for it is the method used whenever we count a (finite) number of things. We count them twice and if the two counts agree then we go no further; if they do not agree, then the items are counted again, etc., until two answers agree. The difference between the situations is that in the case of the circumferential measurements we can postulate a continuous distribution underlying the measurement process, so that a recorded observation is a rounded value of a continuous variable, whereas in the case of counting, the distribution of counts is discrete, with incorrect counts corresponding to actual mistakes.

We shall consider here the continuous case; more specifically, we shall look into the statistical properties of such an estimation procedure when the underlying distribution is normal. It will be shown that an arithmetic mean of two observations from the rounded distribution is almost always superior to the estimator described above for the range of cases considered. The exception occurs only when the rounding interval is so wide and the rounding lattice is so advantageously placed that taking repeat measurements serves simply as a protection against gross errors.

Since the placement of the rounding lattice relative to the true value is usually unknown, the occurrence of the exceptional case, when it occurs, is not known. If the rounding interval is wide and the rounding lattice happens to be *dis*advantageously placed, estimation by duplication is very much inferior to the arithmetic mean of two observations.

Effect of Grouping

The distribution of a single measurement depends on the width of the rounding interval, and also on the placement of the rounding lattice with respect to the true value. We shall assume that measurements are obtained as rounded values of the (continuous) random variable X, which is normally distributed about the true value μ of the property under consideration, with standard deviation σ. The variable X is rounded to the nearest value X_R in a rounding lattice where the interval, centered on X_R, is of length $2Q\sigma$ and the rounding lattice is placed so that the lower boundary of the interval containing μ is at $\mu + D\sigma$ $(-2Q<D\leq 0)$. For simplicity, and with no loss of generality, we let $\mu=0$ and $\sigma=1$. Then the distribution of X_R is given by

$$\text{Prob}\{X_R = x_{Ri}\} = \text{Prob}\{x_{Ri}-Q \leq X \leq x_{Ri}+Q\},$$

$$= \int_{x_{Ri}-Q}^{x_{Ri}+Q} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz,$$

where $x_{Ri} = D + (2i+1)Q$, $i = 0, \pm 1, \pm 2, \ldots$. The distribution of X_R will be completely specified when Q and D are given. We note that the distribution of X_R is discrete, that the mean of X_R is not necessarily zero (it is dependent on the position of the rounding lattice), and that the variance of X_R is always greater than the variance of X. (See Eisenhart, Hastay, and Wallis [1],[1] ch. 4). Only a finite number of values of X_R will have probability realistically different from zero (although, theoretically, there would be an infinite number of them). Strictly speaking, the analysis in this paper treats the normal distribution truncated to

[1] Figures in brackets indicate the literature references at the end of this paper.

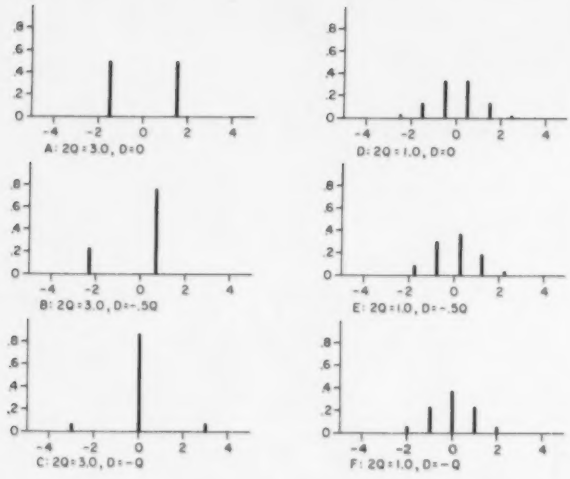

FIGURE 1. *Distributions of X_R for selected values of 2Q and D.*

the interval $(\mu-5\sigma,\ \mu+5\sigma)$. In fact, the numerical results are believed to be correct for the general case.

In studying the properties of this method of estimation by duplication, seven interval lengths were considered: $2Q=3\sigma$, 2σ, 1.5σ, σ, 0.75σ, 0.5σ, and 0.25σ; and 5 positions of the rounding lattice: $D=0, -0.25Q$, $-0.5Q$, $-0.75Q$, and $-Q$. This range of D is sufficient for this study since the distributions of X_R for $-2Q<D<-Q$ are mirror images of the distributions for $-Q<D<0$. Figure 1 illustrates the nature of the distributions from which observations will be taken. Shown are the "best" (i.e., most advantageous) case, where $D=-Q$, the "worst" case, where $D=0$, and an intermediate case, $D=-0.5Q$, for $2Q=3.0$ and $2Q=1.0$. Table 1 shows the mean and variance of the distribution of X_R and the number, m, of rounding intervals (i.e., the number of values of X_R) necessary to cover the range -5σ to 5σ of the normal curve for all cases considered here.

Distribution of T

Having specified the distribution of a single measurement, we may now turn to the estimator under consideration which will be denoted by T. T is the common value of the first two identical measurements in a sequence of measurements. Obviously the sample size N required to obtain identical measurements, as well as their common value T, is a random variable. N can take on the values 2, 3, ... For the purpose of computation, the number of intervals has been limited to m, so that the range of N becomes 2, 3, . . . , $m+1$. Let

$$p_i=\text{Prob}\,\{X_R=x_{Ri}\}=\frac{1}{\sqrt{2\pi}}\int_{x_{Ri}-Q}^{x_{Ri}+Q}e^{-z^2/2}dz.$$

Then, the joint probability that $T=x_{Ri}=t_i$ and $N=n$ is given by

$$P\,\{t_i, n\}=(n-1)!p_i^2\sum\ldots\sum p_jp_k\ldots\ldots p_l, \quad (1)$$

TABLE 1.—*Characteristics of the distribution of X_R: mean, variance, and number of intervals necessary to cover the range -5σ to 5σ of the normal curve*

2Q \ D	0	−0.25Q	−0.5Q	−0.75Q	−Q
3.00	0*	0.0754	0.1065	0.0753	0
	2.2986	2.1318	1.7380	1.3569	1.2027
	4	4	4	5	5
2.00	0	0.0032	0.0046	0.0032	0
	1.3650	1.3557	1.3333	1.3109	1.3016
	6	6	6	6	5
1.50	0	0.0001	0.0001	0.0001	0
	1.1882	1.1880	1.1875	1.1870	1.1868
	8	8	8	7	7
1.00	0	0.0000	0.0000	0.0000	0
	1.0833	1.0833	1.0833	1.0833	1.0833
	10	11	11	11	11
0.75	0	0.0000	0.0000	0.0000	0
	1.0469	1.0469	1.0469	1.0469	1.0469
	14	14	14	15	15
0.50	0	0.0000	0.0000	0.0000	0
	1.0208	1.0208	1.0208	1.0208	1.0208
	20	21	21	21	21
0.25	0	0.0000	0.0000	0.0000	0
	1.0052	1.0052	1.0052	1.0052	1.0052
	40	41	41	41	41

*In each cell of this table, the upper entry is the mean, the middle entry is the variance, and the lower entry is the number of intervals; and "0" means zero exactly, while 0.0000 indicates that the value is zero to at least 4 decimal places.

where the summation is over all $(n-2)$-fold products such that $j<k<\ldots<l$ and $j, k, \ldots, l\neq i$. For $n=2$ and 3 (1) simplifies to

$$P\,\{t_i, 2\}=p_i^2$$

and

$$P\,\{t_i, 3\}=2\,p_i^2(1-p_i) \quad (1a)$$

By summing the quantities (1) and (1a) over all values of N we obtain the distribution of T independent of the value of N, from which we may deduce some of the properties of this measurement procedure. Figures 2A and 3A show the distribution of T for selected values of $2Q$ and D. The means and variances of T,

$$E(T)=\sum_{i=1}^{m}\sum_{n=2}^{m+1}t_iP\,\{t_i, n\}$$

and

$$\text{Var}\,(T)=\sum_{i=1}^{m}\sum_{n=2}^{m+1}t_i^2P\,\{t_i, n\}-[E(T)]^2,$$

for all combinations of Q and D considered are given in table 2. (These and other tabled values are believed to be correct to the accuracy given, taking into account errors due to truncation of the normal distribution and due to rounding during calculation.) Note that the $E(T)$, which are biases of T as an estimator of μ are considerably larger than the $E(X_R)$ (c.f., table 1) for large intervals except in the symmetric cases $(D=0$ and $-Q)$ where $E(T)$ and $E(X_R)$ are identically zero.

Expected values of T conditional on $N=n$, $E(T|n)$, and variances of T conditional on the value of N, Var $(T|n)$, were also calculated, but they are not re-

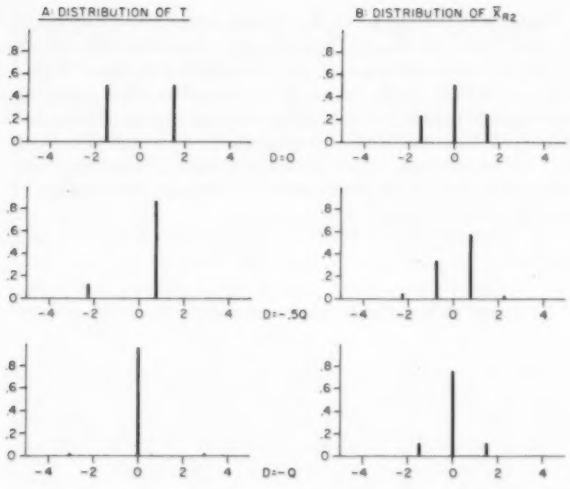

FIGURE 2. *Distributions of T and the corresponding distributions of $\bar{X}_{R2}$ for 2Q = 3.0 and selected values of D.*

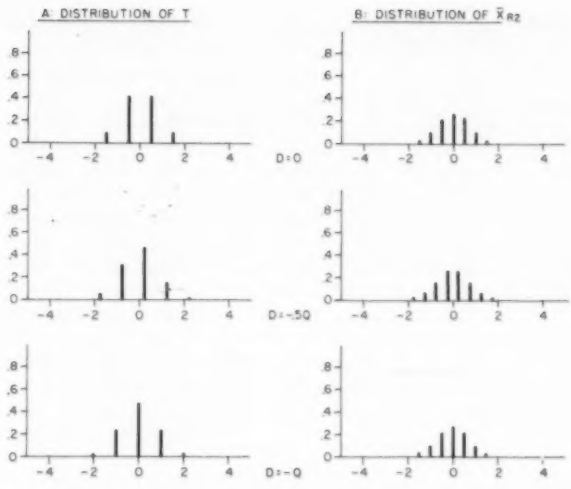

FIGURE 3. *Distributions of T and the corresponding distributions of $\bar{X}_{R2}$ for 2Q = 1.0 and selected values of D.*

ported here. It turned out that, for the non-symmetric positions of the rounding lattice, the conditional biases, $E(T|n)$, are generally larger than the corresponding expected values of X_R and depend not only on the size and placement of the interval but also on the sample size, n. But more disconcerting is the fact that $\mathrm{Var}(T|n)$ increases rather than decreases as n increases. Thus one would be better off with an estimate obtained with $N=2$ than with a larger N. But there is no control over the sample size since N, too, is a random variable. It can be shown that for the maximum value of N, $N=m+1$, the (conditional) distribution of T is identically the distribution of X_R. Thus an estimate based on the maximum sample size is no better than a single observation from the rounded distribution. While $\mathrm{Prob}\{N=m+1\} << 0.0005$, this still is not a happy situation.

TABLE 2. *Values of the mean and variance of the (unconditional) distribution of T for selected values of 2Q and D*

2Q \ D	0	−0.25Q	−0.5Q	−0.75Q	−Q
3.0	0*	0.259	0.351	0.238	0
	2.250	1.850	1.049	0.453	0.258
2.0	0	0.032	0.046	0.032	0
	1.038	0.982	0.848	0.717	0.664
1.5	0	−0.002	−0.002	−0.002	0
	0.743	0.741	0.736	0.732	0.730
1.0	0	0.000	0.000	0.000	0
	0.652	0.652	0.652	0.652	0.652
0.75	0	−0.000	−0.000	−0.000	0
	0.618	0.618	0.618	0.618	0.618
0.5	0	0.000	0.000	0.000	0
	0.589	0.589	0.589	0.589	0.589
0.25	0	0.000	0.000	0.000	0
	0.560	0.560	0.560	0.560	0.560

*In each cell of this table, the upper entry is the mean and the lower entry is the variance; and "0" means zero exactly, while 0.000 indicates that the value is zero to at least 3 decimal places.

By summing the quantities (1) and (1a) over all values of i we obtain the distribution of N. The probabilities,

$$\mathrm{Prob}\{N=n\}=\sum_{i=1}^{m} P\{t_i, n\},$$

and the means and variances of N,

$$E(N)=\sum_{n=2}^{m+1} n\ \mathrm{Prob}\{N=n\}$$

and

$$\mathrm{Var}(N)=\sum_{n=2}^{m+1} n^2\ \mathrm{Prob}\{N=n\}-[E(N)]^2,$$

are given in table 3 for the various distributions of X_R. Probabilities less than 0.0005 are not reported. We see that, for the range of intervals considered here, one would ordinarily expect to take from 3 to 6 observations to obtain an estimate by this procedure. Only when $2Q=3.0$ and the rounding lattice is advantageously placed would one expect to obtain an estimate at $N=2$ — but in these cases the only real reason for taking repeat observations would be as a protection against gross errors.

Comparison of T With $\bar{X}_{R2}$

Since the arithmetic mean is the best estimator of the mean of a normal distribution, it is reasonable to compare T with this fixed sample size estimator. The arithmetic mean of two observations from the distribution of X_R, $\bar{X}_{R2}$, is chosen for comparison with T because, as will be seen below it is almost always a better estimator of μ than is T for the range of Q and D considered here.

TABLE 3. *Values of* Prob {N = n}, *and of the mean and variance of the distribution of* N, *for selected values of 2Q and D*

N \ D	0	−0.25Q	−0.5Q	−0.75Q	−Q
2Q = 3.0					
2	0.497	0.537	0.631	0.722	0.760
3	.499	.457	.356	.258	.217
4	.004	.006	.013	.020	.023
E(N)	2.507	2.470	2.382	2.298	2.264
Var (N)	0.258	0.262	0.262	0.249	0.241
2Q = 2.0					
2	0.457	0.465	0.486	0.507	0.516
3	.478	.464	.429	.394	.379
4	.062	.068	.083	.098	.104
5	.003	.003	.002	.001	.001
E(N)	2.611	2.608	2.601	2.594	2.591
Var (N)	0.379	0.391	0.418	0.445	0.457
2Q = 1.5					
2	0.384	0.385	0.388	0.390	0.391
3	.441	.438	.429	.420	.416
4	.153	.157	.165	.174	.177
5	.021	.020	.018	.016	.015
E(N)	2.812	2.813	2.815	2.817	2.818
Var (N)	0.588	0.590	0.594	0.599	0.600
2Q = 1.0					
2	0.271	0.271	0.271	0.271	0.271
3	.373	.373	.372	.372	.372
4	.252	.252	.253	.253	.254
5	.090	.089	.089	.088	.088
6	.014	.014	.014	.015	.015
7	.001	.001	.001	.001	.001
E(N)	3.206	3.206	3.206	3.206	3.206
Var (N)	0.980	0.979	0.979	0.979	0.979
2Q = 0.75					
2	0.207	0.207	0.207	0.207	0.207
3	.315	.315	.315	.315	.315
4	.271	.271	.271	.271	.271
5	.147	.147	.147	.147	.147
6	.050	.050	.050	.050	.050
7	.010	.010	.010	.010	.010
8	.001	.001	.001	.001	.001
E(N)	3.551	3.551	3.551	3.551	3.551
Var (N)	1.389	1.389	1.389	1.389	1.389
2Q = 0.5					
2	0.140	0.140	0.140	0.140	0.140
3	.234	.234	.234	.234	.234
4	.248	.248	.248	.248	.248
5	.193	.193	.193	.193	.193
6	.114	.114	.114	.114	.114
7	.050	.050	.050	.050	.050
8	.016	.016	.016	.016	.016
9	.004	.004	.004	.004	.004
10	.001	.001	.001	.001	.001
E(N)	4.145	4.145	4.145	4.145	4.145
Var (N)	2.248	2.248	2.248	2.248	2.248
2Q = 0.25					
2	0.070	0.070	0.070	0.070	0.070
3	.129	.129	.129	.129	.129
4	.165	.165	.165	.165	.165
5	.172	.172	.172	.172	.172
6	.155	.155	.155	.155	.155
7	.122	.122	.122	.122	.122
8	.085	.085	.085	.085	.085
9	.052	.052	.052	.052	.052
10	.028	.028	.028	.028	.028
11	.013	.013	.013	.013	.013
12	.006	.006	.006	.006	.006
13	.002	.002	.002	.002	.002
14	.001	.001	.001	.001	.001
E(N)	5.511	5.511	5.511	5.511	5.511
Var (N)	4.957	4.957	4.957	4.957	4.957

Note that the mean of $\overline{X}_{R2}$ is the same as the mean of X_R and that Var $(\overline{X}_{R2}) =$ Var $(X_R)/2$. Figures 2B and 3B show the distributions of $\overline{X}_{R2}$ for $2Q = 3.0$ and 1.0 and $D = 0, -0.5Q$, and $-Q$. We see that the spacing between possible values of the estimates has decreased to half of the width of the original rounding interval so that if the closest value is not obtained, the size of the miss need not be as large as for T and the probability of a large miss is smaller.

For comparing T with X_{R2}, relative efficiency will be used:

$$\text{Efficiency of } T \text{ relative to } \overline{X}_{R2} = \frac{\text{Var}\,(X_R)/2 + [E(X_R)]^2}{\text{Var}\,(T) + [E(T)]^2}$$

(which is usually expressed as a percentage). The relative efficiencies are given in table 4. We see that only in two instances does T show superior behavior over $\overline{X}_{R2}$ – where $2Q = 3.0$, $D = -0.75Q$, and $-Q$ – as indicated by relative efficiencies of 134.3 percent and 232.9 percent respectively. The explanation for these high efficiencies lies in the fact that, in such cases, if an estimate is obtained at $N = 2$ – which, by reference to table 3, happens more than 70 percent of the time – that estimate is almost sure to be the value in the interval containing $\mu = 0$. This leads to a very small variance for the distribution of T conditional on $N = 2$ which offsets the larger variances for $N > 2$. Actually, this effect is also working at the same positions of the rounding lattice for $2Q = 2.0$ and $2Q = 1.5$ but not to the same extent. In all these cases, the conditional variance of T for $N = 2$ is smaller than the variance of an arithmetic mean of 2 observations, so that it *is* possible to obtain better results using estimation by duplication, although these circumstances are limited and not within the control of the observer. For $2Q \leq 1.0$, T is never superior to the mean of 2 observations.

TABLE 4. *Efficiency of* T *relative to the arithmetic mean of two observations for selected values of 2Q and D*

2Q \ D	0	−0.25Q	−0.5Q	−0.75Q	−Q
	%	%	%	%	%
3.0	51.1	55.9	75.1	134.3	232.9
2.0	65.8	69.0	78.4	91.3	98.1
1.5	80.0	80.2	80.6	81.1	81.3
1.0	83.1	83.1	83.1	83.1	83.1
0.75	84.7	84.7	84.7	84.7	84.7
.5	86.7	86.7	86.7	86.7	86.7
.25	89.8	89.8	89.8	89.8	89.8

Comparison of T and $\overline{X}_{R2}$ When the True Value Is Considered To Be a Random Variable

We have seen what results can be expected using estimation by duplication on a single object when the true value of the quantity being estimated is at zero and the rounding lattice is at certain fixed positions relative to zero. If we consider the rounding lattice to be placed at random – which is equivalent to having the rounding lattice centered on zero and assuming that the true values of the quantities are uniformly dis-

tributed between $-Q$ and Q [1, ch 4]—it is apparent that at least for the interval sizes considered here, one would be better off using the mean of two observations as an estimate of the true value rather than the repeated value; for even in the case $2Q=3.0$ one would expect an efficiency of T relative to $\bar{X}_{R2}$ greater than 100 percent only about one-third of the time.

Values for quantities similar to those given in tables 1 through 4 can be calculated under the new assumptions. We note first that T is unbiased under these assumptions, due to symmetry, and that $\bar{X}_{R2}$ is also unbiased for the same reason. Youden, Connor, and Severo [2] have calculated the probability that $N=2$ for intervals of length $2Q=3.0$, 2.0, and 1.0. These probabilities are 0.6296, 0.4860, and 0.2709, respectively. Since estimation by duplication yields results superior to the taking of an arithmetic mean only when $N=2$, as indicated by the conditional variance being smaller than the variance of the mean of two observations, we may use these probabilities to obtain an estimate of the probability of obtaining better results using estimation by duplication. The distribution of T conditional on $N=2$ for $2Q=3.0$ and 2.0 has variance smaller than $\mathrm{Var}(X_R)/2$ for almost all values of D between $-0.5Q$ and $-Q$. Thus if $2Q=3.0$, the probability of obtaining better results with T is approximately $(0.6296)\ (0.5)=0.3148$; and if $2Q=2.0$, the probability is approximately $(0.4860)\ (0.5)=0.2430$. Since the behavior of T is never better than X_{R2} for $2Q=1.0$, the probability of obtaining better results with T is zero.

While the other quantities, such as the mean and variance of the marginal distribution of T, may be of interest, they are difficult to obtain under the new assumptions to any accuracy and would only point more to the fact that estimation by duplication is not a good estimation procedure to use when the underlying distribution is normal. Rough estimates of some of the quantities may be obtained by averaging the appropriate values in the tables given.

From the preceding discussion it follows that when the true value of the quantity to be measured is considered to be uniformly distributed in an interval of length $2Q\sigma$ and measurements of that quantity are normally distributed about its true value with standard deviation σ, then for $2Q \leq 3.0$, the probability is at most 0.3148 that estimation by duplication is better than the arithmetic mean of two observations. For $2Q \leq 1.0$ better results can always be obtained with the arithmetic mean of only two observations.

In conclusion it appears that the practice of taking readings until two identical readings are obtained cannot be justified since the average of the first two readings almost always yields a better estimate of the measured quantity.

The author thanks Churchill Eisenhart for suggesting this investigation and for his guidance, and Joseph M. Cameron and Joan R. Rosenblatt for their helpful suggestions for writing this paper.

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General Application of Youden's Rank Sum Test for Outliers and Tables of One-Sided Percentage Points

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The rank sum test for outliers advanced by W. J. Youden provides a method for detecting if the measurement distribution of any one of a group of objects has a mean significantly different from the rest. This paper discusses a more general application of the rank sum procedure which permits a similar test on other parameters, such as the variance, with the same tables. Tables of the critical values of the extreme rank sum and the corresponding significance levels for one-sided tests are given in this paper to supplement similar tables for two-sided tests already published.

Introduction

W. J. Youden advanced in [1][1] a nonparametric rank sum procedure for detecting whether the distribution of measurements of any one of the rows of a two-way table of measurements had a mean that was noticeably different from the means of the other rows. In Youden's paper approximate 5 percent points for a two-sided test were given. Because of the discrete distribution of the rank sum statistic exact 5 percent points ordinarily do not exist. This technique was further investigated in [2], and there the tables were extended to include approximate 3 percent and 1 percent points along with the actual values of the significance levels.

In all the writings and applications so far, this test has been used to detect an object (or laboratory) whose measurement distribution differs from the distribution of the other objects in its mean or median. However, the test, with the same tables, can often be used to detect an object whose distribution is different with respect to another parameter, for instance, the variance. The purpose of this note is to point out and illustrate the scope of possible applications of Youden's rank sum test, and to give a table of critical regions and significance levels for one-sided tests.

1. The Ordinary and the General Application

The ordinary application of Youden's test as discussed in [1] and [2] is as follows: Suppose there are I objects and J judges. Each judge independently ranks the I objects according to some property. Thus, each object receives a rank, an integer between 1 and I, from each judge, and the J ranks for each object are summed. Then if any object has a rank sum that is extremely high (or low) it can be concluded that the ranks were not assigned just by chance but that the objects tends to rank higher (or lower) than the rest.

In many cases the rankings are assigned on the basis of measurements. That is, there is a measurement for each object-judge combination, and, for a given judge, the object with the highest measurement gets a rank of 1, the object with the next highest gets a rank of 2, etc. (The ranks can be assigned in the reverse order if that is more convenient.) Then if an object gets an extreme rank sum it can be concluded that its measurements tend to be high or low accordingly. This application is actually a test on the means or medians of the measurement distributions for the different objects.

The general application of this technique provides a test on other parameters, and it can be described in the following way. Suppose we have a two-way classification $(I \times J)$ with K_{ij} observations in cell (i, j). The range of the subscripts i and j will be $1, 2, \ldots, I$, and $1, 2, \ldots, J$ respectively throughout the paper. Let the rows correspond to objects being judged and the columns correspond to judges. Now fix attention on the jth column. Suppose the K_{ij} measurements for row i are each distributed with a probability density function, $f_j(x_i; \theta_j(i))$, which depends on a row parameter θ_j that has the value $\theta_j(i)$ for the ith row. Note that each column may have its own distribution, f_j, and its own set of row parameters $\theta_j(i)$.

We wish to test the hypothesis that all the row parameters for any given column are the same,

$$H_0: \theta_j(1) = \theta_j(2) = \ldots = \theta_j(I), \quad j = 1, \ldots, J$$

against the alternative that one of the rows has parameters consistently higher or lower than the rest.

To use the rank sum test there must be an estimate $\hat{\theta}_j(i)$ of the row parameter $\theta_j(i)$ for each cell $(i,j,)$. The distribution of these estimates must be continuous and must satisfy the following conditions.

a) For each j, the joint distribution of $\hat{\theta}_j(1), \hat{\theta}_j(2) \ldots \hat{\theta}_j(I)$ under H_0 is symmetric with respect to the row indexes.

b) The estimates in one column must be independent of the estimates in every other column, i.e., the estimates are mutually independent columnwise. This is the only between-column restriction.

[1] Figures in brackets indicate the literature references at the end of this paper.

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When the above requirements are satisfied, then the rank sum test can be used for H_0 as follows: Within each column compute the estimates $\hat{\theta}_j(1), \ldots, \hat{\theta}_j(I)$ of the row parameters and rank the rows accordingly. Then sum the ranks for each row. An extreme (large or small) value of the rank sum leads to rejection of H_0. The earlier papers, [1] and [2], treated the case $K_{ij} = 1$, in which the single observation in each cell is an estimate of the mean or median of the distribution in that cell. In the general application of the test, instead of ranking the rows (or objects) according to the measurements, we rank them according to estimates computed from the measurements. If (a) and (b) are fulfilled then every permutation of the ranks is equally likely and the original tables for the rank sum test apply. If one of the rows has parameters consistently higher (or lower) than the rest, then that row will tend to receive high (or low) ranks and hence an extreme rank sum. Of course, the efficacy of the test depends on the quality of the estimators, $\hat{\theta}_j(i)$.

The conditions (a) and (b) are sufficient for the tables in [1, 2], and in this paper to apply. It is interesting to note in what situation (a) is fulfilled.

For instance, (a) is implied by

(a′) For each j, $\hat{\theta}_j(1), \ldots, \hat{\theta}_j(I)$ are mutually independent and identically distributed.

Thus, consider the common situation where (1) $K_{ij} = K_j$, that is, there is the same number of measurements in each cell in the same column, (2) the measurements are all independent between cells, and (3) $\hat{\theta}_j(i)$ is computed only from the measurements in cell (i, j) and is computed in the same way for every cell. Then (a′) holds and the test can be used. It is not necessary that there be the same number of measurements per cell from column to column. For example, suppose the test is on medians and $\hat{\theta}_j(i)$ is the sample median for each cell. Then it is permissible to have, say, three measurements per cell in the first column, one per cell in the second column, four per cell in the third, etc.

There are other situations where the $\hat{\theta}_j(i)$ are not computed from the measurements in cell (i, j) only. For example, suppose $K_{ij} = 1$ and $\theta_j(i)$ is the absolute deviation of the measurement in (i, j) from the column mean. Then (a′) does not hold because the $\hat{\theta}_j(i)$'s are not independent; however, (a) may still hold and then the test can be used. In this example, if the measurements in the cells are identically and independently distributed under H_0, then (a) is fulfilled because the joint distribution of the $\hat{\theta}_j(i)$ for each j, although not independent, is still symmetric with respect to the row indices.

Example 1, Test on Means: Consider the usual test on the means. If $K_{ij} = 1$, and x_{ij} is the measurement in cell (i, j), then $\hat{\theta}_j(i) = x_{ij}$ is the estimate of the ith row mean for column j. If $K_{ij} = K_j > 1$, then the cell average or the cell median would be used for $\hat{\theta}_j(i)$.

Example 2, Test on Variance: Suppose the measurements have the same distribution, row to row, except possibly for the variances σ^2_{ij}. We wish to test

$$H_0: \sigma^2_{ij} = \sigma^2_{2j} = \ldots = \sigma^2_{ij}, \quad \text{for } j = 1, \ldots, J$$

Here we need $K_{ij} = K_j \geq 2$, then we can take $\hat{\theta}_j(i) = s^2_{ij}$, the sample within-cell variance. In this case the estimates are independent, hence (a′) holds, and the rank sum test can be applied. A one-sided test might be appropriate here.

An alternative test for equality of variances could be based on the within-cell sample ranges, or any other dispersion measure.

Example 3, Test on Accuracy: Suppose $K_{ij} = 1$ and we wish to discover whether any row is substantially less accurate than the others. By this is meant that its means differ from the other row means, or it has larger variances, or any combination of these such that its measurements tend to differ from the measurements of the other rows.[2]

The measurements x_{ij} are assumed to have two parameter distributions with means μ_{ij} and variances σ^2_{ij}. We test

$$H_0: \mu_{1j} = \mu_{2j} = \ldots = \mu_{Ij}, \quad \sigma_{1j} = \sigma_{2j} = \ldots = \sigma_{Ij}, \qquad j = 1, \ldots, J$$

against the alternative described above. The rank sum test can be used by letting

$$\hat{\theta}_j(i) = |x_{ij} - x_{\cdot j}|$$

where $x_{\cdot j} = \frac{1}{I}\sum_{i=1}^{I} x_{ij}$, and then ranking the rows accordingly. If some row's measurements deviate consistently from the column averages, it will receive an extreme rank sum. Notice that the statistics $\hat{\theta}_j(i)$ satisfy condition (a), hence the regular rank sum tables apply. A one-sided test is appropriate here.

It is interesting to see just what the statistics $\hat{\theta}_j(i)$ do estimate. First, it clearly does not matter whether we rank by using $\hat{\theta}_j(i)$ or $\hat{\theta}^2_j(i)$ so we look at the latter. It turns out that

$$E[\hat{\theta}^2_j(i)] = E[(x_{ij} - x_{\cdot j})^2] = \frac{1}{I^2}\left\{(I-1)^2\sigma^2_{ij} + \left[\sum_{i'}(\mu_{ij} - \mu_{i'j})\right]^2 + \sum_{i'}\sigma^2_{i'j}\right\}$$

where $\sum_{i'}$ means the sum over $i' \neq i$.

Consider the special case where all but one of the rows are the same. Let

$$\mu_{2j} = \mu_{3j} = \ldots = \mu_{Ij} = \mu_j$$

$$\sigma^2_{2j} = \sigma^2_{3j} = \ldots \sigma^2_{Ij} = \sigma^2_j$$

and

$$\mu_{1j} = \mu_j + \alpha_j$$

$$\sigma^2_{1j} = \sigma^2_j + \beta_j$$

[2] This application of Youden's test was first suggested by Robert B. Dean of the Borden Chemical Co. In a recent paper [3] T. W. Lashof proposed a "center outward" ranking procedure of a similar nature. He suggested that, say, for n odd, the median measurement be awarded a "rank" of 1, the next larger and the next smaller both a "rank" of 2 and so on. No tables are available for Lashof's technique.

then

$$E[(x_{ij}-x_{.j})^2]=\left(\frac{I-1}{I}\right)\sigma_j^2+\left(\frac{I-1}{I}\right)^2(\alpha_j^2+\beta_j)$$

and for $i \neq 1$,

$$E(x_{ij}-x_{.j})^2=\left(\frac{I-1}{I}\right)\sigma_j^2+\frac{1}{I^2}(\alpha_j^2+\beta_j).$$

Thus

$$E[\hat{\theta}_j^2(1)]-E[\hat{\theta}^2_j(2)]=\frac{I-2}{I}(\alpha_j^2+\beta_j)$$

Hence, if either the $\beta_j > 0$ and they are large, or if the α_j's are large, then $\hat{\theta}_j(1)$ will tend to be larger than $\hat{\theta}_j(2), \ldots, \hat{\theta}_j(I)$ and the first row will receive an extreme rank sum.

TABLE 1.—*Tables for a one-sided rank sum test at nominal significance levels of 1, 3, and 5 percent*
$\alpha = \Pr_{min} \leq [J + R)] = \Pr_{max} \geq [IJ - R)]$

No. of objects I	$I=3$						$I=4$					
	0.01		0.03		0.05		0.01		0.03		0.05	
No. of judges J	R	α	R	α	R	α	R	α	R	α	R	α
3											0	0.062
4			0	0.037	0	0.037	0	0.0156	0	0.016	1	.078
5	0	0.0123	0	.012	1	.074	0	.0039	1	.023	2*	.082
6	0	.0041	1	.029	2*	.115	1	.0068	2	.027	3*	.082
7	1	.0110	1	.011	2	.049	2	.0088	3	.029	4*	.079
8	1	.0041	2	.021	3	.072	3	.0101	4	.030	5*	.074
9	2	.0084	3	.032	4*	.095	4	.0108	5	.029	6	.069
10	3	.0140	4	.045	5*	.119	5	.0110	6	.028	7	.063
11	3	.0060	4	.021	5	.059	6	.0110	7	.026	8	.058
12	4	.0094	5	.029	6*	.074	7	.0107	8	.025	9	.053
13	5	.0136	6	.038	7*	.090	8	.0103	9	.023	10	.048
14	5	.0062	6	.018	7	.047	9	.0098	10	.021	11	.043
15	6	.0088	7	.024	8	.057	10	.0092	12	.039	13*	.072

No. of objects I	$I=5$						$I=6$					
	0.01		0.03		0.05		0.01		0.03		0.05	
No. of judges J	R	α	R	α	R	α	R	α	R	α	R	α
3			0	0.040	1*	0.160			0	0.028	1*	0.111
4	0	0.0080	1	.040	2*	.120	0	0.0046	1	.023	2	.069
5	1	.0096	2	.034	3*	.090	1	.0046	3	.043	4*	.097
6	2	.0090	3	.027	4	.067	3	.0108	4	.027	5	.059
7	3	.0077	4	.021	5	.050	4	.0071	6	.037	7*	.072
8	4	.0063	6	.038	7*	.078	6	.0107	7	.023	8	.045
9	6	.0126	7	.028	8	.057	7	.0068	9	.028	10	.051
10	7	.0096	8	.021	9	.042	9	.0089	11	.032	12	.056
11	8	.0073	10	.031	11	.058	11	.0109	13	.036	14	.060
12	10	.0117	11	.023	12	.043	13	.0127	14	.022	15	.038
13	11	.0088	13	.032	14	.056	14	.0080	16	.024	17	.040
14	13	.0128	14	.024	15	.042	16	.0091	18	.026	19	.042
15	14	.0096	16	.031	17	.053	18	.0100	20	.027	21	.043

No. of objects I	$I=7$						$I=8$					
	0.01		0.03		0.05		0.01		0.03		0.05	
No. of judges J	R	α	R	α	R	α	R	α	R	α	R	α
3			0	0.020	1*	0.082	0	0.0156	0	0.016	1	0.062
4	1	0.0146	2	.044	3*	.102	1	.0098	2	.029	3	.068
5	2	.0087	3	.023	4	.052	3	.0137	4	.031	5	.061
6	4	.0125	5	.028	6	.055	4	.0064	6	.028	7	.052
7	5	.0067	7	.029	8	.054	7	.0131	8	.024	9	.043
8	7	.0078	9	.029	10	.052	9	.0116	11	.035	12	.058
9	9	.0083	11	.028	12	.048	11	.0099	13	.029	14	.046
10	11	.0085	13	.026	14	.044	13	.0083	16	.036	17	.056
11	13	.0084	15	.024	16	.040	16	.0114	18	.029	19	.044
12	15	.0080	18	.036	19	.055	18	.0092	21	.034	22	.050
13	17	.0076	20	.032	21	.049	21	.0115	23	.027	25	.056
14	20	.0115	22	.028	23	.043	23	.0092	26	.030	27	.044
15	22	.0105	24	.025	26	.055	26	.0110	29	.034	30	.048

No. of objects I	$I=9$						$I=10$					
	0.01		0.03		0.05		0.01		0.03		0.05	
No. of judges J	R	α	R	α	R	α	R	α	R	α	R	α
3	0	0.0123	0	0.012	1	0.049	0	0.0100	1	0.040	2*	0.100
4	1	.0069	2	.021	3	.048	2	.0150	3	.035	4*	.070
5	3	.0085	5	.038	6*	.070	4	.0126	5	.025	6	.046
6	5	.0078	7	.029	8	.051	6	.0092	8	.030	9	.050
7	8	.0121	10	.036	11	.059	9	.0114	11	.032	12	.050
8	10	.0091	12	.026	13	.042	11	.0076	14	.032	15	.048
9	13	.0114	15	.029	16	.045	14	.0081	17	.030	18	.044
10	15	.0082	18	.031	19	.047	17	.0082	20	.028	22	.058
11	18	.0094	21	.033	22	.047	20	.0081	23	.026	25	.051
12	21	.0103	24	.033	25	.047	24	.0114	27	.033	28	.045
13	24	.0109	27	.033	28	.046	27	.0105	30	.029	32	.054
14	27	.0113	30	.033	31	.045	30	.0096	33	.026	35	.047
15	30	.0115	33	.032	34	.044	33	.0087	37	.031	39	.054

No. of objects I	$I=11$						$I=12$					
	0.01		0.03		0.05		0.01		0.03		0.05	
No. of judges J	R	α	R	α	R	α	R	α_U	R	α_U	R	α_U
3	0	0.0083	1	0.033	2*	0.082	0	0.007	1	0.028	2	0.069
4	2	.0113	3	.026	4	.052	2	.009	3	.020	4	.040
5	4	.0086	6	.032	7	.054	5	.012	6	.022	8	.062
6	7	.0107	9	.031	10	.049	8	.012	10	.032	11	.049
7	10	.0110	12	.028	13	.043	11	.011	13	.026	15	.057
8	13	.0104	15	.025	17	.054	14	.009	17	.030	18	.043
9	16	.0094	19	.031	20	.044	18	.011	21	.032	22	.045
10	19	.0083	22	.026	24	.050	21	.008	25	.034	26	.045
11	23	.0105	26	.030	28	.055	25	.009	29	.033	30	.045
12	26	.0088	30	.033	31	.044	29	.010	33	.033	35	.057
13	30	.0102	33	.026	35	.047	33	.010	37	.032	39	.054
14	34	.0115	37	.028	39	.048	37	.010	41	.030	43	.051
15	37	.0093	41	.030	43	.050	41	.010	45	.029	47	.047

No. of objects I	$I=13$						$I=14$					
	0.01		0.03		0.05		0.01		0.03		0.05	
No. of judges J	R	α_U	R	α_U	R	α_U	R	α_U	R	α_U	R	α_U
3	0	0.006	1	0.023	2	0.059	0	0.005	1	0.020	2	0.051
4	2	.007	4	.032	5	.057	3	.013	4	.025	5	.046
5	5	.009	7	.028	8	.045	6	.012	8	.033	9	.052
6	8	.008	11	.033	12	.050	9	.009	12	.034	13	.050
7	12	.010	15	.035	16	.050	13	.010	16	.032	17	.046
8	16	.011	19	.035	20	.048	17	.010	20	.029	22	.054
9	19	.008	23	.033	24	.045	21	.009	25	.034	26	.046
10	23	.008	27	.031	29	.055	25	.008	29	.029	31	.050
11	28	.011	31	.028	33	.049	30	.010	34	.032	36	.053
12	32	.011	36	.033	38	.055	34	.009	39	.033	41	.054
13	36	.010	40	.029	42	.048	39	.010	43	.027	46	.055
14	40	.009	45	.033	47	.052	44	.011	48	.028	51	.054
15	45	.010	49	.029	51	.045	48	.009	53	.028	55	.044

See footnote at end of table.

TABLE 1.—*Tables for a one-sided rank sum test at nominal significance levels of 1, 3, and 5 percent*—Continued

$\alpha = P(r_{min} \leq J + R) = P(r_{max} \geq IJ - R)$

No. of objects I	$I=15$					
No. of judges J	0.01		0.03		0.05	
	R	α_U	R	α_U	R	α_U
3	0	0.004	1	0.018	2	0.044
4	3	.010	5	.037	6	.062
5	6	.009	8	.025	10	.059
6	10	.010	12	.024	14	.051
7	14	.010	17	.030	19	.057
8	18	.009	22	.034	23	.045
9	23	.011	26	.027	28	.047
10	27	.009	31	.028	33	.046
11	32	.009	36	.027	38	.045
12	37	.010	41	.027	44	.053
13	42	.010	47	.032	49	.050
14	47	.010	52	.030	54	.046
15	52	.009	57	.028	59	.043

*In order to avoid duplication in the tables these values of R do not give α's as close as possible to the nominal significance levels listed. They are already given in the column immediately adjacent, hence the tabulated R gives another still higher (or lower) significance level.

2. Tables for a One-Sided Test

In both the ordinary application of the rank sum test for outliers and in some of the examples given in sec. 1, one-sided tests may be appropriate. Table 1 gives approximate 0.01, 0.03, and 0.05 one-sided probability points for the extreme rank sum. It is tabulated in much the same way as the table of two-sided probability points given in [2].

The table can be described as follows: Let I be the number of objects ranked and J be the number of judges. Then the smallest that a rank sum could be is J and the largest it could be is IJ. Let r_i be the rank sum of the ith object, and let

$$r_{max} = \max_{1 \leq i \leq I} r_i,$$

$$r_{min} = \min_{1 \leq i \leq I} r_i$$

Let R be an integer and α be a probability such that

$$\alpha = P(r_{min} \leq J + R) = P(r_{max} \geq IJ - R)$$

under the null hypothesis that all rankings are equally likely.

Table 1 gives, for $3 \leq I \leq 15, 3 \leq J \leq 15$, the values of R yielding significance levels, α, as close as possible to the nominal significance levels of 0.01, 0.03, 0.05. In small print next to each R is given the corresponding value of α.

For example, suppose $I=8$ and $J=6$. Then all the rank sums must be between 6 and 48. From table 1 we see that $R=4$ yields $\alpha=0.0064$. Thus, a one-sided test for large rank sums at a significance level 0.0064 can be made by rejecting the hypothesis when $r_{max} \geq 48-4=44$. That is, if any rank sum is 44 or more, it is rejected as too large. Similarly, the corresponding test for small rank sums would be to reject whenever $r_{min} \leq 6+4=10$.

From the same line of the table we see that a critical region of $r_{max} \geq 42$ has a significance level of $\alpha=0.028$, and a critical region of $r_{max} \geq 41$ has $\alpha=0.052$. Because of the discrete nature of the rank sum statistic, $\alpha=0.0064$, 0.028, and 0.052 are the closest significance levels possible to 0.01, 0.03, and 0.05, respectively.

Table 1 can be thought of as an abbreviated table of the distribution function of the largest (or smallest) rank sum. In several places the values of R yielding α's closest to 0.03 and to 0.05 are the same. To avoid duplication and to offer a wider choice of significance levels in the table, the value of R belonging to both 0.03 and 0.05 is given in the 0.03 column and the next higher value of R is given in the 0.05 column even though it doesn't "belong" there. The same procedure was followed whenever possible and when there was a duplication in the 0.01 and the 0.03 column. These places in the table are marked with an "*".

3. Computation of the Table

The significance level α for any given R cannot be computed easily, but bounds, α_U and α_L, such that $\alpha_L \leq \alpha \leq \alpha_U$ can be. Using eqs (2), (3), and (6) of [2] α_U and α_L were computed for $I \leq 11$. For the 0.01 (0.03) level, α_U and α_L always agree to four (three) decimal places, hence the values in the table for α are exact to the four (three) decimal places given.

For the 0.05 level α_U and α_L did not always agree when rounded to three places although

$$\alpha_U - \alpha_L < 0.0005$$

all the time. The rounded value for α_L was chosen since α_L is a better approximation to α.

For $I \geq 12$, α_L was not computed, hence α_U, not α, is given in the table. From the computations performed for [2] for large I it can safely be assumed that α_U agrees with α to 3 decimal places in the 0.01 and 0.03 column and that the α_U's given in the 0.05 column are in error by at most 0.002. Hence α_U can be used as a good approximation to α.

4. References

[1] W. J. Youden, Ranking laboratories by round-robin tests. Materials Research and Standards **3**, 9–13 (1962).

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A Generalization of Rennie's Inequality[1]

A. J. Goldman

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A generalization is given of Rennie's recent extension of the Kantorovich inequality. The result is used in an alternate proof of the sharp upper bounds for ratios of weighted means obtained by Cargo and Shisha.

1. Introduction

For $n > 1$, consider a sequence of $2n$ positive numbers

$$x_1, \ldots, x_n; w_1, \ldots, w_n$$

not necessarily distinct, and such that

$$0 < m \leq x_i \leq M; \; w_1 + \ldots + w_n = 1$$

where M and m are constants. A well-known and useful inequality due to Kantorovich [3][2] asserts that under these circumstances

$$\left(\sum w_i x_i\right) \cdot \left(\sum w_i x_i^{-1}\right) \leq (M+m)^2/4Mm. \tag{1}$$

Recently Rennie [4] gave a simple proof of an inequality which in turn yields an especially elegant derivation of (1). Also recently, Cargo and Shisha [1] obtained a general inequality which gives sharp upper bounds for the ratios of weighted means,

$$\left(\sum w_i x_i^s\right)^{1/s} \left(\sum w_i x_i^r\right)^{-1/r} \leq \{r^{-1}(q^r - 1)\}^{-1/s} \{s^{-1}(q^s - 1)\}^{1/r} \{(s-r)^{-1}(q^s - q^r)\}^{(1/s)-(1/r)}, \tag{2}$$

where $q = M/m$, valid for real s and r satisfying[3]

$$s > r, \; sr \neq 0. \tag{3}$$

The inequality (1) is obtained by setting $s = 1$ and $r = (-1)$.

Our purpose here is to see how Rennie's proof of (1) might be adapted to a proof of (2). Although the generalization is not quite as elegant as the original special case (the sources of the extra complexity are noted at appropriate points in the text), the resulting alternate proof of (2) and generalization of Rennie's inequality seem worth recording.

The notation employed will be appropriate to the continuous version of the results (which specializes to the discrete version); the "weights" w_i of (1) and (2) are replaced by a fixed measure space of unit total measure, the numbers x_i are replaced by a real-valued measurable function x satisfying $0 < m \leq x \leq M$ almost everywhere, and sums like $\Sigma\, w_i x_i$ or $\Sigma\, w_i x^s$ are replaced by integrals

[1] The author is indebted to K. Goldberg (NBS Numerical Analysis Section) for helpful discussions.

[2] Figures in brackets indicate the literature references at the end of this paper.

[3] The limiting cases with $sr = 0$ are also treated in [1], but will be omitted here.

$I(x)$ or $I(x^s)$ taken over the entire space. It will be convenient to *normalize* by replacing x with the measurable function $y = x/m$, which satisfies

$$1 \leq y \leq q = M/m$$

almost everywhere. Our generalization of Rennie's inequality (which after "denormalization" is the special case $s = 1$, $r = (-1)$) asserts under hypotheses (3) that

$$q^s - q^r \leq (q^s - 1)I(y^r) - (q^r - 1)I(y^s) \qquad \text{if } sr > 0, \tag{4a}$$

$$q^s - q^r \geq (q^s - 1)I(y^r) - (q^r - 1)I(y^s) \qquad \text{if } sr < 0. \tag{4b}$$

For real $r \neq 0$, the rth order *mean* of x (with *weights* given by the measure) is defined as

$$M_r(x) = \{I(x^r)\}^{1/r}.$$

In terms of these means we can define the quotients

$$Q_{sr}(x) = M_s(x)/M_r(x).$$

Since the quotients are unchanged by normalization, the continuous version of the Cargo-Shisha result (2) can be written

$$Q_{sr}(y) \leq \{r^{-1}(q^r - 1)\}^{-1/s}\{s^{-1}(q^s - 1)\}^{1/r}\{(s-r)^{-1}(q^s - q^r)\}^{(1/s)-(1/r)}. \tag{5}$$

This and (4) will be proved in section 3, after disposing of some preliminaries in the following section 2.

Two natural directions for possible further investigations suggest themselves. The first is to seek a generalized inequality involving more than two powers of y. The second is to consider integrals $I\{f(y)\}$ for functions f other than power functions.

2. Preliminaries

The basis of the analysis turns out to be the famous inequality between the arithmetic and geometric means[4], which asserts for $a > 0$ and $b > 0$ that

$$a^\alpha b^{1-\alpha} \leq \alpha a + (1-\alpha)b \qquad \text{if } 0 < \alpha < 1, \tag{6a}$$

$$a^\alpha b^{1-\alpha} \geq \alpha a + (1-\alpha)b \qquad \text{if } \alpha < 0, \tag{6b}$$

with equality if and only if $a = b$. The form (6b) is obtained from the more familiar form (6a) by applying the latter to

$$a' = a^{1-\alpha}, b' = b^{1-\alpha}, \alpha' = -\alpha/(1-\alpha)$$

and then multiplying the result through by $(1-\alpha)a^\alpha$ and rearranging terms. Rennie [4] required only the special case $\alpha = 1/2$ which is particularly easy to prove[5], but it is known [2] that the general case of (6a) can be deduced from this special case by elementary algebraic methods plus a simple limiting process to handle irrational α.

As a first application of (6), we introduce the function[6]

$$\varphi(u) = \varphi(u; t, \tau) = (u^{\tau+t} - 1)/(u^t - 1) \qquad (t\tau > 0)$$

and prove it to be strictly increasing for $u > 0$ if $t > 0$ and $\tau > 0$, but strictly decreasing for $u > 0$

[4] Cargo and Shisha [1] show how their inequality yields another proof of this classical one.
[5] Expand the left-hand side of $(a^{1/2} - b^{1/2})^2 \geq 0$ and rearrange terms.
[6] We give $\varphi(u)$ its limiting value $(\tau + t)/t$ at $u = 1$.

if $t<0$ and $\tau<0$. For this purpose, calculate for $u\neq 1$

$$\varphi'(u)=(u^t-1)^2u^{t-1}(\tau+t)\{\tau(\tau+t)^{-1}u^{\tau+t}+t(\tau+t)^{-1}-u^\tau\}.$$

To prove that the last factor is positive, so that (as desired) the derivative has the same sign as $\tau+t$, we need only apply (6a) with

$$a=u^{\tau+t},\ b=1,\ \alpha=\tau(\tau+t)^{-1}.$$

Rennie [4] required only the obvious special case that

$$\varphi(u;1,1)=u+1$$

is increasing.

Assuming (3) as always, we now show for $1\leq Y\leq q$ that

$$(q^s-1)Y^r-(q^r-1)Y^s\geq q^s-q^r \qquad \text{if } sr>0, \tag{7a}$$

$$(q^s-1)Y^r-(q^r-1)Y^s\leq q^s-q^r \qquad \text{if } sr<0, \tag{7b}$$

with equality if and only if $Y=1$ or $Y=q$. Clearly equality holds in the indicated cases, so we assume $1<Y<q$ and prove strict inequality. For $sr>0$, assertion (7a) is equivalent if $s>r>0$ to

$$(q^s-1)/(q^r-1)=\varphi(q;r,s-r)>\varphi(Y;r,s-r)=(Y^s-1)/(Y^r-1),$$

and if $r<s<0$ to

$$(q^r-1)/(q^s-1)=\varphi(q;s,r-s)<\varphi(Y;s,r-s)=(Y^r-1)/(Y^s-1).$$

For $sr<0$, assertion (7b) is equivalent to

$$(Y^s-1)/(Y^r-1)<(q^s-1)/(q^r-1)$$

and thus to

$$(Y^{s-r}-Y^{-r})/(Y^{-r}-1)=\varphi(Y;-r,s)-1<\varphi(q;-r,s)-1=(q^{s-r}-q^{-r})/(q^{-r}-1).$$

So (7) follows from the properties of $\varphi(u)$ proved in the last paragraph.

3. Proofs of Results

The values Y of the normalized function y obey $1\leq Y\leq q$ almost everywhere, so that our generalized Rennie inequality (4) is obtained from (7) simply by integrating over the measure space. From the discussion of (7) it follows that equality occurs in (4) if and only if y differs from 1 and q only on a set of measure zero. Denote the measure by μ, the inverse transformation of y by y^{-1} (to be distinguished from the function $1/y$ obtained from y^r for $r=(-1)$), and define

$$\theta=\mu\{y^{-1}(q)\}; \tag{8}$$

then the condition for equality in (4) can be written

$$1-\theta=\mu\{y^{-1}(1)\}. \tag{9}$$

For $sr>0$ rewrite (4a) as

$$\{-r(s-r)^{-1}\}\{r^{-1}(q^r-1)I(y^s)\}+\{s(s-r)^{-1}\}\{s^{-1}(q^s-1)I(y^r)\}\geq(s-r)^{-1}(q^s-q^r), \tag{10a}$$

and for $sr < 0$ rewrite (4b) as

$$\{-r(s-r)^{-1}\}\{r^{-1}(q^r-1)I(y^s)\}+\{s(s-r)^{-1}\}\{s^{-1}(q^s-1)I(y^r)\}\leq(s-r)^{-1}(q^s-q^r). \tag{10b}$$

We shall apply the arithmetic-mean geometric-mean inequality (6), with [7]

$$a=r^{-1}(q^r-1)I(y^s),\ b=s^{-1}(q^s-1)I(y^r),\ \alpha=-r(s-r)^{-1}, \tag{11}$$

to the left-hand side of (10). If $s>r>0$ we apply (6b) to (10a) and obtain

$$\{r^{-1}(q^r-1)I(y^s)\}^{-r/(s-r)}\{s^{-1}(q^s-1)I(y^r)\}^{s/(s-r)}\geq(s-r)^{-1}(q^s-q^r), \tag{12}$$

which when raised to the negative power $(s-r)/(-sr)$ yields

$$\{r^{-1}(q^r-1)I(y^s)\}^{1/s}\{s^{-1}(q^s-1)I(y^r)\}^{-1/r}\leq\{(s-r)^{-1}(q^s-q^r)\}^{(1/s)-(1/r)}. \tag{13}$$

If $sr<0$ we apply (6a) to (10b) to obtain the reverse of (12), which when raised to the positive power $(s-r)/(-sr)$ yields (13) again. Finally, if $r<s<0$ we shift from (11) to

$$a=s^{-1}(q^s-1)I(y^r),\ b=r^{-1}(q^r-1)I(y^s),\ \alpha=s(s-r)^{-1} \tag{14}$$

and apply (6b) to (10a) to obtain (12), which when raised to the negative power $(s-r)/(-sr)$ yields (13). So in all cases we have (13), a rearrangement of the Cargo-Shisha inequality (5).

Equality can occur in (5) if and only if it occurs in (4), so that (9) holds, and *also* $a=b$ in (11) and (14). Substitution of (9) into this last condition leads to

$$r^{-1}(q^r-1)\{\theta q^s+(1-\theta)\cdot 1\}=s^{-1}(q^s-1)\{\theta q^r+(1-\theta)\cdot 1\}$$

or equivalently

$$\theta=\theta(q;s,r)=(s-r)^{-1}\{r(q^r-1)^{-1}-s(q^s-1)^{-1}\}, \tag{15}$$

the same condition for equality obtained in [1]. To show that the bounds in (5) are sharp, it suffices to prove that

$$0<\theta(q;s,r)<1. \tag{16}$$

The left-hand inequality is equivalent for $s>r>0$ to

$$(q^s-1)/(q^r-1)=\varphi(q;r,s-r)>\varphi(1;r,s-r)=s/r,$$

for $r<s<0$ to

$$(q^r-1)/(q^s-1)=\varphi(q;s,r-s)<\varphi(1;s,r-s)=r/s,$$

and for $sr<0$ to

$$(q^{s-r}-q^{-r})/(q^{-r}-1)=\varphi(q;-r,s)-1>\varphi(1;-r,s)-1=s/(-r).$$

The right-hand inequality in (16) can be rewritten

$$r/(q^{-r}-1)>s/(q^{-s}-1).$$

This is equivalent for $s>r>0$ to

$$(q^{-s}-1)/(q^{-r}-1)=\varphi(q;-r,r-s)<\varphi(1;-r,r-s)=s/r,$$

[7] From now on we exclude the trivial case $q=1$ and assume $q>1$.

for $r < s < 0$ to

$$(q^{-r}-1)/(q^{-s}-1) = \varphi(q; -s, s-r) > \varphi(1; -s, s-r) = r/s,$$

and for $sr < 0$ to

$$(q^{-s}-1)/(q^{-r}-1) = 1-\varphi(q; r, -s) > 1-\varphi(1; r, -s) = s/r.$$

Thus (16) is verified.

Added in proof: Rennie's inequality turns out to be equivalent to an apparently stronger one given by Diaz and Metcalf [5] in the same year, and our generalization is in turn equivalent to one announced previously by Diaz and Metcalf [6]. The proofs of these (non-trivial) equivalences will be presented in a forthcoming paper by Diaz, Metcalf, and the writer.

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On the Asymptotic Joint Normality of Quantiles From a Multivariate Distribution[1]

Lionel Weiss[2]

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A simple proof is given of the asymptotic joint normality of sample quantiles from a multivariate population, under very mild conditions. The joint cumulative distribution function of the quantiles is studied, rather than the joint probability density function.

1. Introduction

Suppose $X_1, \ldots, X_k$ are jointly distributed random variables, with absolutely continuous cumulative distribution function $F(x_1, \ldots, x_k)$. $F_i(x_i)$ and $f_i(x_i)$ denote respectively the marginal cumulative distribution function and marginal density function for X_i. $F_{i,j}(x_i, x_j)$ denotes the joint marginal cumulative distribution function for X_i, X_j.

Let $b_1, \ldots, b_k$ be fixed quantities, each in the open interval $(0, 1)$. Suppose n independent observations are taken on the vector $(X_1, \ldots, X_k)$. Denote by $Y_i(n)$ the b_ith sample quantile of the n observed values of X_i. That is, $Y_i(n)$ is equal to that one of the n observed X_i's such that exactly $[nb_i]$ other X_i's are below it, where $[nb_i]$ denotes the largest integer contained in nb_i. We can ignore the possibility of ties, which occur with probability zero.

We assume that the equation $F_i(y) = b_i$ has a unique solution in y, which we denote by u_i $(i = 1, \ldots, k)$, and that $f_i(x_i)$ is continuous and positive at $x_i = u_i$. Define

$$Z_i(n) = \sqrt{n}\, f_i(u_i) \frac{Y_i(n) - u_i}{\sqrt{b_i(1-b_i)}}.$$

We shall show that as n increases, the joint cumulative distribution function of $Z_1(n), \ldots, Z_k(n)$ approaches the k-variate normal cumulative distribution function with zero means, unit variances, and the covariance between the ith and jth variables $(i \neq j)$ given by

$$\frac{F_{i,j}(u_i, u_j) - b_i b_j}{\sqrt{b_i b_j (1-b_i)(1-b_j)}}.$$

This was proved by Mood [2] for the special case $b_i = \ldots = b_k = 1/2$, and by Siddiqui [3] for the case $k = 2$ and under the assumption that the first and second partial derivatives of $F(x_1, x_2)$ are continuous at u_1, u_2 and the joint probability density function is positive there. Siddiqui conjectured that the result holds for general k, and sketched a proof, assuming that the partial derivatives of $F(x_1, \ldots, x_k)$ of order k are continuous at $u_1, \ldots, u_k$. Both Mood and Siddiqui used the joint probability density function of $Z_1(n), \ldots, Z_k(n)$, and the resulting computations were lengthy. Below we give a simple proof, under less restrictive conditions. The proof uses the joint cumulative distribution function of $Z_1(n), \ldots, Z_k(n)$.

2. Proof of Asymptotic Normality

Fix a point $(z_1, \ldots, z_k)$ in $(x_1, \ldots, x_k)$-space. Define

$$y_i(n) = u_i + \frac{z_i \sqrt{b_i(1-b_i)}}{f_i(u_i)\sqrt{n}}.$$

Use the point $(y_1(n), \ldots, y_k(n))$ as the origin of the usual system of orthogonal axes in $(x_1, \ldots, x_k)$-space. These axes define 2^k "octants" in the space. We assume that these octants are labeled in some definite order, which will be held fixed. For example, when $k = 2$, we have four quadrants, rather than "octants," which we assume are labeled in the conventional way. Let $N_j(n)$ denote the number of the n observed k-dimensional vectors which fall within the jth octant. Because of the continuity of $F(x_1, \ldots, x_k)$, we can ignore the possibility that one or more of the observed vectors will fall on the boundary of an octant. The quantities $N_j(n)$ $(j = 1, \ldots, 2^k)$ have a multinomial distribution with parameters n, $p_1(n), \ldots, p_{2^k}(n)$, where $p_j(n)$ is the probability assigned to the jth octant by the distribution function $F(x_1, \ldots, x_k)$. Clearly, as n increases $p_j(n)$ approaches the probability assigned by $F(x_1, \ldots, x_k)$ to the jth octant determined by axes through the point $(u_1, \ldots, u_k)$. We denote this limiting value of $p_j(n)$ by p_j.

Let the symbol $\sum_j^{(i)}$ denote the operation of summing over all those values of j such that in the jth octant, the ith coordinate is greater than $y_i(n)$. There are 2^{k-1} such values of j. For example, when $k = 2$,

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using the conventional labeling of the quadrants, $\sum_j^{(1)}$ would sum over $j=1, 4$, and $\sum_j^{(2)}$ would sum over $j=1, 2$. Define $L_i(n)$ as $\sum_j^{(i)} N_j(n)$. Then the event $\{Z_i(n) < z_i\}$ is the same as the event $\{L_i(n) < n-[nb_i]\}$. Therefore

$$P[Z_1(n) < z_1, \ldots, Z_k(n) < z_k]$$
$$= P[L_1(n) < n-[nb_1], \ldots, L_k(n) < n-[nb_k]]. \quad (1)$$

Next, we define

$$M_j(n) = \frac{N_j(n) - np_j(n)}{\sqrt{np_j(n)}} \qquad (j=1, \ldots, 2^k).$$

It is well known that as n increases, the joint distribution of $M_1(n), \ldots, M_{2^k}(n)$ approaches a normal distribution with zero means, variance of $M_j(n)$ equal to $(1-p_j)$, and covariance between $M_h(n)$ and $M_j(n)$ equal to $-\sqrt{p_h p_j}$. (See Cramer [1][3] p. 318.) We can write

$$L_i(n) = \sum_j^{(i)} [np_j(n) + \sqrt{np_j(n)}\, M_j(n)] \quad (2)$$

Clearly

$$\sum_j^{(i)} np_j(n) = n[1 - F_i(y_i(n))]$$
$$= n\left[1 - F_i\left(u_i + \frac{z_i}{\sqrt{n}\, f_i(u_i)} \sqrt{b_i(1-b_i)}\right)\right]$$
$$= n(1-b_i) - \sqrt{n}\, z_i \sqrt{b_i(1-b_i)} - n\, g_i(n) \quad (3)$$

where $\sqrt{n}\, g_i(n)$ approaches zero as n increases. Also,

$$n - [nb_i] = n(1-b_i) + c_i \quad (4)$$

where $|c_i| < 1$. Substituting (2), (3), and (4) in (1), and rearranging terms, we find

$$P[Z_1(n) < z_1, \ldots, Z_k(n) < z_k]$$
$$= P\left[\frac{\sum_j^{(i)} \sqrt{p_j(n)} M_j(n)}{\sqrt{b_i(1-b_i)}} < z_i + \frac{\sqrt{n} g_i(n)}{\sqrt{b_i(1-b_i)}} + \frac{c_i}{\sqrt{n}\sqrt{b_i(1-b_i)}};\ i=1, \ldots, k\right]. \quad (5)$$

Since $p_j(n)$ approaches p_j as n increases, and $\sqrt{n} g_i(n)$ and $c_i/\sqrt{n}$ both approach zero as n increases, (5) shows that the joint asymptotic distribution of $Z_1(n), \ldots, Z_k(n)$ is the same as the joint asymptotic distribution of

$$\frac{\sum_j^{(1)} \sqrt{p_j} M_j(n)}{\sqrt{b_1(1-b_1)}}, \ldots, \frac{\sum_j^{(k)} \sqrt{p_j} M_j(n)}{\sqrt{b_k(1-b_k)}} \quad (6)$$

if this latter asymptotic joint distribution is absolutely continuous, which it is, since it is a nonsingular k-variate normal distribution. Therefore the asymptotic joint distribution of $Z_1(n), \ldots, Z_k(n)$ is normal, with zero means and variances and covariances the same as in the asymptotic joint distribution of the variables listed in (6). In computing these variances and covariances, we can look at just two of the variables in (6) at a time, and there is no loss of generality in looking at the first two. In doing this, we can assume that we are taking n observations on (X_1, X_2), ignoring $(X_3, \ldots, X_k)$. This reduces the problem to the case $k=2$, where $F_{1,2}(x_1, x_2)$ is the joint distribution of the basic variables, and we have four quadrants. Using the conventional labeling of the quadrants, we have

$$\frac{\sum_j^{(1)} \sqrt{p_j} M_j(n)}{\sqrt{b_1(1-b_1)}} = \frac{\sqrt{p_1} M_1(n) + \sqrt{p_4} M_4(n)}{\sqrt{b_1(1-b_1)}}$$
$$\frac{\sum_j^{(2)} \sqrt{p_j} M_j(n)}{\sqrt{b_2(1-b_2)}} = \frac{\sqrt{p_1} M_1(n) + \sqrt{p_2} M_2(n)}{\sqrt{b_2(1-b_2)}}$$
$$p_1 = 1 - b_1 - b_2 + F_{1,2}(u_1, u_2)$$
$$p_2 = b_1 - F_{1,2}(u_1, u_2)$$
$$p_4 = b_2 - F_{1,2}(u_1, u_2).$$

Using these facts, and the variances and covariances in the joint asymptotic distribution of $M_1(n)$, $M_2(n)$, $M_3(n)$, $M_4(n)$ given above, a straightforward computation shows that the variances in the asymptotic distribution of $Z_1(n)$ and $Z_2(n)$ are equal to unity, and the covariance is equal to

$$\frac{F_{1,2}(u_1, u_2) - b_1 b_2}{\sqrt{b_1 b_2 (1-b_1)(1-b_2)}}.$$

This completes the proof of the theorem.

3. Extensions

It is clear that the method above also proves the joint asymptotic normality for the case where we include several different quantiles for each X_i. In this case, we use several sets of axes, giving a collection of regions determined by the intersections of the octants defined by the various sets of axes. For the jth such region, we define $N_j(n)$ as the number of observed vectors falling in it, and proceed as above.

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Fitting $y=\beta x$ When the Variance Depends on x

John Van Dyke[1]

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This paper presents some results concerning the selection of a method for estimating the slope of a straight line through the origin. For fitting the line $y=\beta x$ when the variance of y is proportional to x^p, it is well known that the best estimate of β depends on p. In practice, however, only integer values of p would be convenient to work with. One of the estimators appropriate for $p=0,1,2$ would probably be used if the value of p were in fact fractional, or if it were only approximately known. This paper provides some guides for choosing the best among these estimators in a particular situation.

Formulas for the best estimators of β and their variances are given. Estimators of β appropriate for integer values of p are compared in the case when p is not integral, but is known, and in the case when p is only approximately known. Estimation of the variances of estimators of β is considered. Finally, some results are given on the effect of the spacing of the x values on the comparison of the estimators.

This paper presents some results concerning the selection of a method for estimating the slope of a straight line through the origin. For fitting the line $y=\beta x$ when the variance of y is proportional to x^p, it is well known that the best estimator of β depends on p. In practice, however, only integer values of p would be convenient to work with. One of the estimators appropriate for $p=0,1,2$ would probably be used if the value of p were in fact fractional, or if it were only approximately known. This paper provides some guides for choosing the best one in a particular situation.

The line $y=\beta x$ is to be fitted to observed points $(x_1, y_1), \ldots, (x_n, y_n)$, where $x_i>0$. The values of x are assumed to be known without error. For fixed x_i, the corresponding y_i, is an observed value of the random variable Y_i with mean βx_i and variance $V(Y_i)=\sigma^2 x_i^p$, $p \geq 0$.

Let B_p denote the "best," i.e., the minimum variance unbiased linear, estimator of β when $V(Y_i)=\sigma^2 x_i^p$, and let $V_p(B_p)$ be its variance. Denote by $V_p(B_r)$ the variance of the estimator B_r when $V(Y_i)=\sigma^2 x_i^p$ and $p \neq r$. When r is restricted to the integers it will be represented by k.

Section 1 contains formulas for estimators of β and for the variances of these estimators. Comparisons of B_k and B_{k+1} are considered in sections 2 and 3, with numerical calculations given for the case of equally spaced values of x. Section 2 is a comparison of B_k and B_{k+1} (k integral) when p is known and $k<p<k+1$. Values of $p^*=p_n(k, k+1)$ are given such that when $k \leq p<p^*$, B_k is better than B_{k+1}, and when $p^*<p \leq k+1$, then B_{k+1} is better. In section 3, B_k and B_{k+1} are compared when it is known that $k<p<k+1$, but the value of p is unknown. Comparison of the *relative efficiencies* of these estimators shows that B_k is better than B_{k+1} for $k<p<k+1$. In section 4, an estimator, t_r, of the variance of B_r is considered. It is an unbiased estimator of $V_p(B_r)$ whenever $r=p$ and for all p when $r=2$. It is positively biased when $p<r<2$, and negatively biased when $r<p<2$. Section 5 contains some results on the effect of the spacing of the x values on the comparison of estimators.

Methods for fitting a straight line when the variance depends on x are treated in a general way in, for example, Brownlee's chapter on linear regression.[2] The present paper provides further details relating to the choice of weights, for a particular case of the problem considered in Brownlee's section 11.14, "Weighted Regression through the Origin." The methods employed here could be adapted for consideration of other cases.

1. Formulas for B_p and $V_p(B_p)$

Let $Y_1, Y_2, \ldots, Y_n$ be mutually independent random variables corresponding to a set of exact positive values $x_1, x_2, \ldots, x_n$, with expectations

$$E(Y_i)=\beta x_i,$$

and let the variances $V(Y_i)$ be proportional to x_i^p, i.e., let

$$V(Y_i)=\sigma^2 x_i^p, \qquad p \geq 0.$$

Then the best estimator B_p of β is the value of B_p that minimizes the sum of weighted squared residuals,

$$\sum_{i=1}^{n} \frac{(y_i-B_p x_i)^2}{\sigma^2 x_i^p}.$$

[1] Present address: Bureau of Scientific Research, Food and Drug Administration, Washington, D.C.

[2] K. A. Brownlee, Statistical Theory and Methodology in Science and Engineering, New York, John Wiley & Sons, Inc., 1960 (Chapter 11).

The estimator is

$$B_p = \frac{\sum_{i=1}^{n} y_i x_i^{1-p}}{\sum_{i=1}^{n} x_i^{2-p}}. \qquad (1\text{–}1)$$

Its variance is given by

$$V_p(B_p) = \frac{\sigma^2}{\sum_{i=1}^{n} x_i^{2-p}}. \qquad (1\text{–}2)$$

Table 1 gives formulas for B_p and $V_p(B_p)$ for $p=0$, 1 and 2, when there are n points: (x_1, y_1), (x_2, y_2), . . . , (x_n, y_n).

TABLE 1.—*Formulas for* B_p *and* $V_p(B_p)$

p	B_p	$V_p(B_p)$
0	$\frac{\sum_{i=1}^{n} x_i y_i}{\sum_{i=1}^{n} x_i^2}$	$\frac{\sigma^2}{\sum_{i=1}^{n} x_i^2}$
1	$\frac{\sum_{i=1}^{n} y_i}{\sum_{i=1}^{n} x_i} = \frac{\bar{y}}{\bar{x}}$	$\frac{\sigma^2}{\sum_{i=1}^{n} x_i}$
2	$\frac{1}{n}\sum_{i=1}^{n}\left(\frac{y_i}{x_i}\right) = \left(\overline{\frac{y}{x}}\right)$	$\frac{\sigma^2}{n}$

If there are m values of y for each x, then y_i in the above formulas is replaced by y_{ij} with $j=1, 2, \ldots, m$, and the denominator of each estimate and variance has the additional factor m; e.g., for $p=1$,

$$B_1 = \frac{\sum_{j=1}^{m}\sum_{i=1}^{n} y_{ij}}{m\sum_{i=1}^{n} x_i}$$

and

$$V_1(B_1) = \frac{\sigma^2}{m\sum_{i=1}^{n} x_i}$$

It has been assumed for convenience that all x_i are positive. In general, one might have both positive and negative values of x. The above formulas are easily modified for use when the variance depends on the absolute value of x,

$$V(Y_i) = \sigma^2 |x_i|^p.$$

Alternatively, the formulas in table 1 may be used directly provided every pair (x_i, y_i) having $x_i < 0$ has been replaced by $(-x_i, -y_i)$.

2. Comparison of B_k and B_{k+1} (k Integral) When $k<p<k+1$

Since numerical calculation of B_p and its variance is not easy when p is not an integer, the question arises as to which estimator, B_k or B_{k+1}, is the "better" estimator when p is known but lies between the integers k and $k+1$. Regardless of the true value of p, B_k is an unbiased estimator of β. Thus the question of which of the two is the "better" reduces to the question of which one has the smaller variance. This can be answered if we can find a value p^* such that $k<p^*<k+1$ and

$$V_p(B_k) < V_p(B_{k+1}), \quad k \leq p < p^*$$

$$V_p(B_k) > V_p(B_{k+1}), \quad p^* < p \leq k+1.$$

Here $V_p(B_k)$ denotes the variance of B_k calculated under the assumption that $V(Y_i)=\sigma^2 x_i^p$. In general,

$$V_p(B_r) = \frac{\sigma^2 \sum_{i=1}^{n} x_i^{2-2r+p}}{\left[\sum_{i=1}^{n} x_i^{2-r}\right]^2}. \qquad (2\text{–}1)$$

The value of p^* may be found by setting $V_p(B_k)$ equal to $V_p(B_{k+1})$, and then solving for p.

Values of p^* have been found for the case of equally spaced positive values of x; that is, $x_i=ic$, $i=1, 2, \ldots, n$, and c is any positive constant. The line $y=\beta x$ is then fitted to the points (c, y_1), $(2c, y_2)$, . . . , (nc, y_n). Let $p_n(k, k+1)$ denote the solution of the equation $V_p(B_k)=V_p(B_{k+1})$ in the interval $k<p<k+1$. Values of $p_n(k, k+1)$ were found for $k=0$, 1, and for various values of n. For small n $(2 \leq n \leq 10)$, $p_n(k, k+1)$ was found by graphical methods. The exact values of $p_n(k, k+1)$ for $k=0$, $n=2$ and $k=1$, $n=2$, 3, were also obtained directly by solving the equation $V_p(B_k)=V_p(B_{k+1})$. These results agree well enough with the graphical results so that it is believed that the results obtained graphically are correct to within one unit in the second decimal place. The value for $n=100$ was obtained by approximate methods. For $n \to \infty$, $p_\infty(k, k+1)$ was found by letting n tend to infinity in the equation $V_p(B_k)=V_p(B_{k+1})$ and then solving the resulting equation for p. The values of $p_n(k, k+1)$ are given in table 2. Since the range of $p_n(k, k+1)$ is comparatively small over all values of n, we state a convenient rule:

When	*Use*
$0<p<0.6$	B_0
$0.6<p<1.6$	B_1
$1.6<p<2+?$	B_2

Table 3 contains formulas for the estimators B_k ($k=0, 1, 2$) for equally spaced x and gives their variances $V_p(B_k)$ for $p=0, 1, 2$.

Figure 1 and figure 2 give, for $n=3$ and $n=10$ respectively, values of $V_p(B_p)$ and $V_p(B_k)$ for p rang-

TABLE 2. *Values of* $p_n(k, k+1)$ *for equally spaced* x

n	$k=0$	$k=1$
2	0.5406	1.5146
3	.56	1.5350
4	.57	1.55
5	.57	1.56
6	.58	1.57
7	.58	1.58
8	.59	1.58
9	.59	1.59
10	.59	1.59
100	.599+	1.64
∞	.600---	1.66---

TABLE 3. *Variances* $V_p(B_k)$ *and slopes* B_k *for equally spaced* x

p	$k=0$	$k=1$	$k=2$
0	$\frac{6\sigma^2}{n(n+1)(2n+1)c^2}$	$\frac{4\sigma^2}{n(n+1)^2c^2}$	$\frac{\sigma^2\sum_{i=1}^{n} i^{-2}}{n^2c^2}$
1	$\frac{9\sigma^2}{c(2n+1)^2}$	$\frac{2\sigma^2}{cn(n+1)}$	$\frac{\sigma^2\sum_{i=1}^{n} i^{-1}}{cn^2}$
2	$\frac{6(3n^2+3n-1)\sigma^2}{5n(n+1)(2n+1)}$	$\frac{2(2n+1)\sigma^2}{3n(n+1)}$	$\frac{\sigma^2}{n}$
B_k	$\frac{6\sum_{i=1}^{n} iy_i}{cn(n+1)(2n+1)}$	$\frac{2\sum_{i=1}^{n} y_i}{cn(n+1)}$	$\frac{\sum_{i=1}^{n} i^{-1}y_i}{cn}$

ing from 0.0 to at most 3.0 and $k=0$, 1, and 2. Figures such as these (on a larger scale) were used to obtain the values of $p_n(k, k+1)$, $2 \le n \le 10$. For convenience, the values of c and σ^2 were chosen equal to one. It is evident from the two figures, and it also can be shown mathematically, that $V_p(B_p)$ and $V_p(B_k)$ are increasing functions of p when all values of x are at least equal to one.

3. Relative Efficiency of B_k, B_{k+1} When $k<p<k+1$

Suppose it is only known that the value of p is in the interval $(k, k+1)$. Which estimator of β, B_k, or B_{k+1}, is then the "better" of the two? One way to answer the question is to use the idea of *relative efficiency*. The efficiency of an estimator T_1 of θ relative to an estimator T_2 of θ is defined to be

$$\text{R.E.}\,(T_1|T_2) = 100\,\frac{E(T_2-\theta)^2}{E(T_1-\theta)^2},$$

expressed as a percentage. If R.E. $(T_1|T_2) > 100$ percent, T_1 may be considered the better estimator of θ. If $E(T_1)=E(T_2)=\theta$, the relative efficiency is the ratio of the variances,

$$\text{R.E.}\,(T_1|T_2) = 100\,\frac{V(T_2)}{V(T_1)}.$$

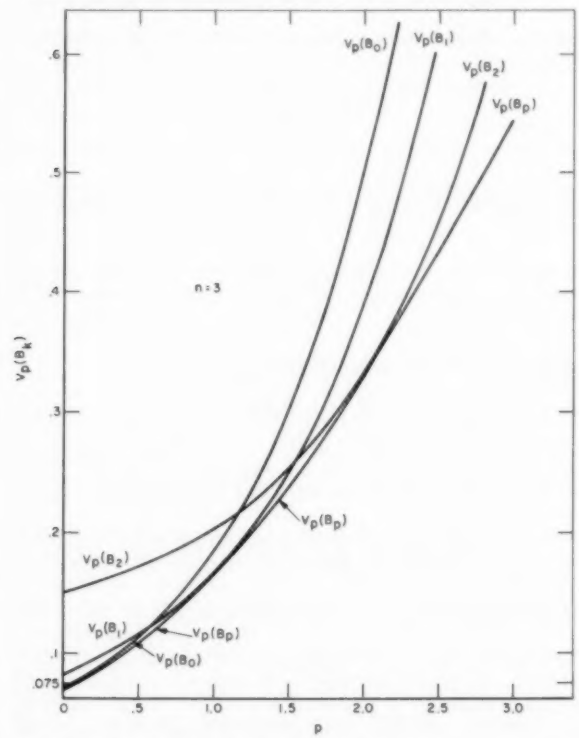

FIGURE 1. *Variances* $V_p(B_k)$, $V_p(B_p)$ *for* $c=1$, $\sigma^2=1$, $n=3$.

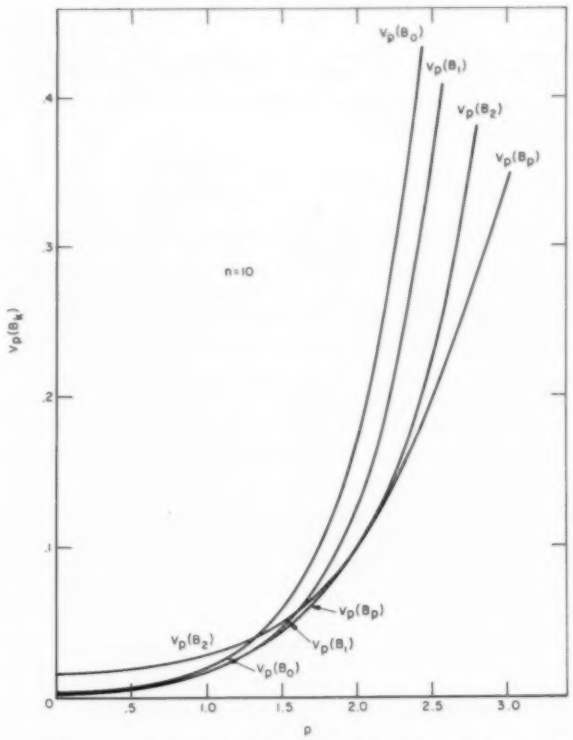

FIGURE 2. *Variances* $V_p(B_k)$, $V_p(B_p)$ *for* $c=1$, $\sigma^2=1$, $n=10$.

To compare B_k with B_{k+1}, we consider the least advantageous situation for each: (1) let B_k be the estimator when in fact $p=k+1$, and (2) let B_{k+1} be the estimator when $p=k$. Under these conditions the best estimator can be found by comparing two particular relative efficiencies:

(1) the efficiency of B_k relative to B_{k+1}, assuming $p=k+1$,

$$\text{R.E.}\,(B_k|B_{k+1})=\frac{V_{k+1}(B_{k+1})}{V_{k+1}(B_k)}, \tag{3-1}$$

(2) the efficiency of B_{k+1} relative to B_k, assuming $p=k$,

$$\text{R.E.}\,(B_{k+1}|B_k)=\frac{V_k(B_k)}{V_k(B_{k+1})}. \tag{3-2}$$

Both of these efficiencies are less than 100 percent, because the numerator is the variance of the estimator whose variance is minimum. However, if one of these relative efficiencies is larger than the other, then the estimator whose variance is the denominator of this larger ratio may be considered the better of the two estimators in the sense that it is less inefficient under the worst possible circumstances. A comparison between such pairs of relative efficiencies was made for the case of equally spaced x when $k=0$ and $k=1$. This comparison of the relative efficiencies showed that for both values of k,

$$\text{R.E.}\,(B_k|B_{k+1}) > \text{R.E.}\,(B_{k+1}|B_k)$$

whenever $n \geq 2$. Thus, B_0 is better than B_1 when $0 \leq p \leq 1$ and B_1 is better than B_2 when $1 \leq p \leq 2$.

Figure 3 and figure 4 show for $n=3$ and $n=10$ respectively values of

$$\text{R.E.}\,(B_l|B_k)=100\left[\frac{V_p(B_k)}{V_p(B_l)}\right],$$

for all p in the interval 0.0 to 3.0 and k and l equal to some combinations of 0, 1, and 2. To verify the conclusion stated in the previous paragraph, notice for example that (in both figures) V_0/V_1 at $p=0$ is smaller than V_1/V_0 at $p=1$; that is, the minimum of

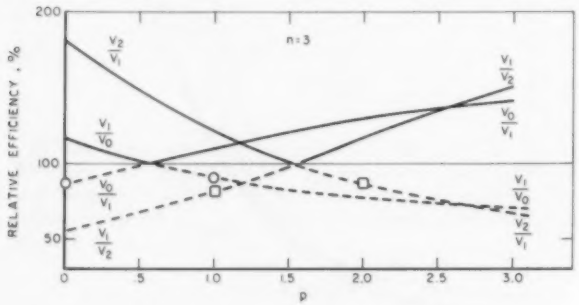

FIGURE 3. *Relative efficiency curves for* n = 3.

R.E. $(B_l|B_k)=100\left[\frac{V_p(B_k)}{V_p(B_l)}\right]=\frac{V_k}{V_l}$

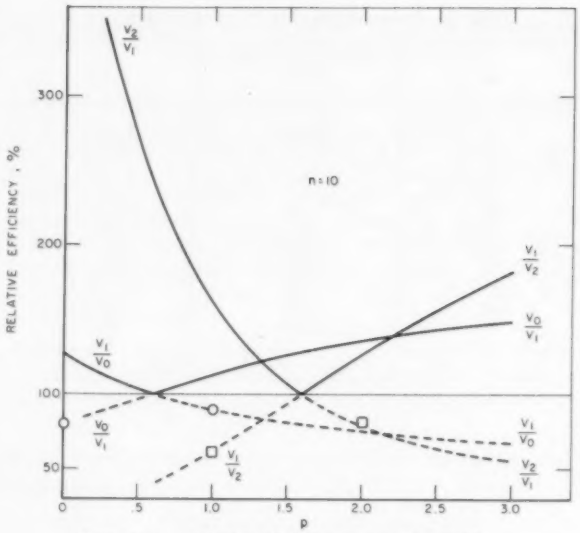

FIGURE 4. *Relative efficiency curves for* n = 10.

R.E. $(B_l|B_k)=100\left[\frac{V_p(B_k)}{V_p(B_l)}\right]=\frac{V_k}{V_l}$

R.E. $(B_1|B_0)$ is less than the minimum of R.E. $(B_0|B_1)$ in the interval $0 \leq p \leq 1$.

4. Estimation of the Variance of B_r

It has been shown that the variance of estimator B_r is

$$V_r(B_r)=\frac{\sigma^2}{\sum_{i=1}^{n} x_i^{2-r}}.$$

Let t_r denote the following estimator for the variance of B_r:

$$t_r=\frac{S_r^2}{\sum_{i=1}^{n} x_i^{2-r}}, \tag{4-1}$$

where S_r^2 is an estimator of σ^2,

$$S_r^2=\frac{1}{n-1}\sum_{i=1}^{n}\frac{(y_i-B_r x_i)^2}{x_i^r}, \qquad r \geq 0,$$

or, in a form more convenient for computation

$$S_r^2=\frac{1}{n-1}\left[\sum_{i=1}^{n}\frac{y_i^2}{x_i^r}-B_r^2\sum_{i=1}^{n}x_i^{2-r}\right].$$

When $V(Y_i)=\sigma^2 x_i^p$, the expectation of t_r is

$$E_p(t_r)=\frac{\sigma^2}{n-1}\left[\frac{\sum_{i=1}^{n}x_i^{p-r}}{\sum_{i=1}^{n}x_i^{2-r}}-\frac{\sum_{i=1}^{n}x_i^{2-2r+p}}{\left(\sum_{i=1}^{n}x_i^{2-r}\right)^2}\right]. \tag{4-2}$$

When $p=r$, t_r is an unbiased estimator of $V_r(B_r)$. When $p \neq r$, one would like the expectation, $E_p(t_r)$, to be $V_p(B_r)$ as given in eq (2–1), but this is not usually the case. However, when $r=2$, t_r is an unbiased estimator; i.e., $E_p(t_2)=V_p(B_2)$ for all p. The bias of estimator t_r is the difference, $E_p(t_r)-V_p(B_r)$, and this quantity is positive or negative depending upon the values of p and r. In general, for positive values of x which are not all equal, the bias of estimator t_r is positive if $p<r<2$, and it is negative if $r<p<2$; i.e.,

$$E_p(t_r) > V_p(B_r), p < r < 2,$$

$$E_p(t_r) < V_p(B_r), r < p < 2.$$

A sketch of the proof of this result follows.

$$E_p(t_r)-V_p(B_r)=\frac{\sigma^2}{n-1}\frac{\left(\sum_{i=1}^{n} x_i^{p-r}\sum_{i=1}^{n} x_i^{2-r}-n\sum_{i=1}^{n} x_i^{2-2r+p}\right)}{\left(\sum_{i=1}^{n} x_i^{2-r}\right)^2}, \tag{4–3}$$

which can also be written as

$$\frac{\sigma^2}{\left(\sum_i x_i^{2-r}\right)^2}\frac{1}{n-1}\sum_{i<j}(x_i^{p-r}-x_j^{p-r})(x_j^{2-r}-x_i^{2-r}).$$

Consider the product $(x_i^{p-r}-x_j^{p-r})(x_j^{2-r}-x_i^{2-r})$; and take $x_1 \leq x_2 \leq \ldots \leq x_n$, but not all equal.

(1) If $p<r<2$, the signs of the two factors are (+) (+) for all pairs x_i, x_j for which $x_i \neq x_j$. The sum of such terms is positive and hence the bias is positive.

(2) If $r<p<2$, the signs are (−) (+) for all pairs x_i, x_j where $x_i \neq x_j$, and in this case the bias is negative.

Thus the estimator t_r is likely to overestimate the variance $V_p(B_r)$ if r is such that $p<r<2$, and it is likely to underestimate $V_p(B_r)$ if $r<p<2$.

For the case of equally spaced x, the effect of differences between p and r is shown in figure 5, where the *relative bias* of estimator t_r,

$$\text{R.B.}=100\left[\frac{E_p(t_r)-V_p(B_r)}{V_p(B_r)}\right], \tag{4–4}$$

is plotted against $p, 0 \leq p \leq 2$, for $r=k=0$, 1 and $n=3, 10$. This quantity is independent of σ^2 and of the constant c.

Figure 5 illustrates the possibility that the estimated variance of B_k may be substantially biased if k is not close to p. For example, suppose that $n=10$ and the value of p is unknown, but it is known to be in the interval (0, 1). If $k=0$, t_0 would most likely underestimate $V_p(B_0)$, for the relative bias ranges downward from 0.0 percent (at $p=0$) to about −33 percent (at $p=1$). On the other hand, if $k=1$, t_1 would be likely to overestimate $V_p(B_1)$, and the bias in this case could be as high as 68 percent (if $p=0$) of $V_p(B_1)$.

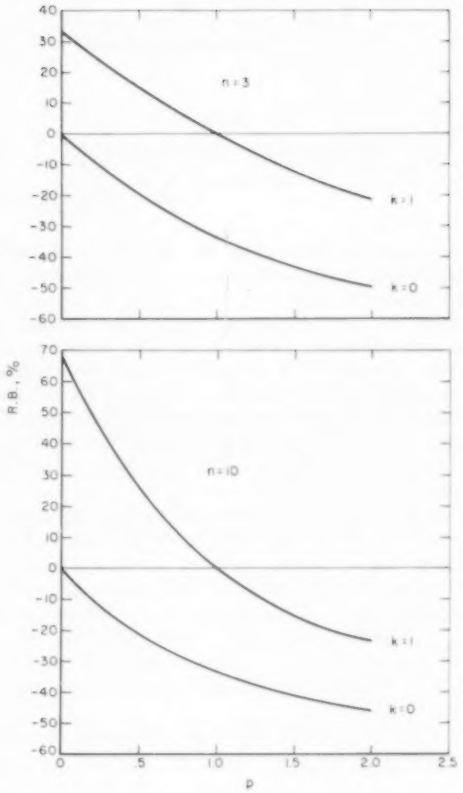

FIGURE 5. *Relative bias of estimator* t_r, *when* $r=k=0, 1$.

R.B. $= 100\,[E_p(t_r)-V_p(B_k)]/V_p(B_k)$

5. Effect of the Spacing of the x Values on the Comparison of the Estimators

Up to this point emphasis has been on the case where the x values were n in number and were equally spaced on the interval $(0, nc)$; that is, $x_i=ic$, $i=1, 2, \ldots, n$; and to each x there corresponded a single y.

Now suppose that instead of n different x's, there are $N=n/m$ equally spaced x's on the interval $(0, nc)$, and m y's corresponding to each x; that is, the line $y=\beta x$ is fitted to the n points, $(x_i, y_{ij})=(mic, y_{ij})$, $i=1, 2, \ldots, N, j=1, 2, \ldots, m$.

Of course, if $m=1$, we are back to the case of n x's and a single y for each, so it is a special case of this more general form. For the more general case, the estimator B_p and variances $V_p(B_p)$ and $V_p(B_r)$ have the following forms:

$$B_p=\frac{\sum_{j=1}^{m}\sum_{i=1}^{N} y_{ij} i^{1-p}}{cm^2\sum_{i=1}^{N} i^{2-p}}, \tag{5–1}$$

$$V_p(B_p)=\frac{\sigma^2}{c^{2-p}m^{3-p}\sum_{i=1}^{N} i^{2-p}}, \qquad (5\text{–}2)$$

$$V_p(B_r)=\frac{\sigma^2\sum_{i=1}^{N} i^{2-2r+p}}{c^{2-p}m^{3-p}\left[\sum_{i=1}^{N} i^{2-r}\right]^2}. \qquad (5\text{–}3)$$

In comparing estimators B_k and B_{k+1} in section 2, the equation, $V_p(B_k)=V_p(B_{k+1})$, was solved for p, and the solutions, $p_n(k, k+1)$, are given in table 2. The solutions of this equation when the variances are of the general form, eq (5–3), are also found in table 2. It is the number of x values, N, that determines the solution, and so the solutions are found by substituting the value of N for the value n of the table. For example, if $n=15$ and $m=3$, then $N=15/3=5$, and the solutions are those corresponding to $n=5$, i.e., 0.57 and 1.56.

In section 3, the relative efficiency of estimators was used to show that estimator B_k was better than estimator B_{k+1} under certain circumstances. The same result is obtained for the more general case. In addition, any combination of n and m for which N is constant gives the same relative efficiency (eq (3–1) or (3–2)). Thus, figure 4 is identical to those obtained for all pairs n and m such that $N=n/m=10$.

The estimator t_r (introduced in sec. 4) of the variance $V_r(B_r)$ can here be written in the form

$$t_r=\frac{S_r^2}{c^{2-r}m^{3-r}\sum_{i=1}^{N} i^{2-r}}, \qquad (5\text{–}4)$$

where

$$S_r^2=\frac{1}{(n-1)c^r m^r}\left[\sum_{j=1}^{m}\sum_{i=1}^{N}\frac{y_{ij}^2}{i^r}-\frac{\left(\sum_{j=1}^{m}\sum_{i=1}^{N} y_{ij}i^{1-r}\right)^2}{m\sum_{i=1}^{N} i^{2-r}}\right].$$

The results given in section 4 on the sign of the bias of estimator t_r apply in the general case. However, the magnitude of the relative bias (R.B.) of t_r is affected by the choice of N and m for fixed $n=mN$.

Table 4 gives the relative bias for $n=10$, and shows that it is smallest for $N=1$ (which holds for all n). Table 4 can also be used to calculate the relative bias of t_r for other values of n.

From section 4, the R.B. for n observations is given by eq (4–4), using eqs (4–3) and (2–1). Let the formula obtained from eq (4–4) with $x_i=i$ be denoted by $G(n)$. For the general case, the R.B. for $N=n/m$ equally spaced x values is found to be

$$\text{R.B.}\,(N, n)=\left(\frac{n}{n-1}\right)\left(\frac{N-1}{N}\right)G(N).$$

Table 4 gives R.B. (N, n) for $n=10$, and also approximately R.B. (N, n) for other values of n, where the approximation should be fairly good if N and n are large. More precisely,

$$\text{R.B.}\,(N, hn)=\frac{h(n-1)}{hn-1}\,\text{R.B.}\,(N, n).$$

TABLE 4. *Values of* $V_p(B_k)$, $E_p(t_k)$, *and R.B.* (N, n) *for* $n=10$, $\sigma^2=1$, $c=1$, *and* $N=n/m$

p	N	$k=0$			$k=1$			$k=2$*
		$V_p(B_k)$	$E_p(t_k)$	R.B.	$V_p(B_k)$	$E_p(t_k)$	R.B.	$V_p(B_k)$
0	10	0.002597	0.002597	0	0.003306	0.005550	68	0.01550
	5	.002273	.002273	0	.002778	.003920	41	.00732
	2	.001600	.001600	0	.001778	.002025	14	.00250
	1	.001000	.001000	0	.001000	.00100	0	.00100
1	10	.02041	.01361	−33	.01818	.01818	0	.0293
	5	.01860	.01309	−30	.01667	.01667	0	.0228
	2	.01440	.01173	−19	.01333	.01333	0	.0150
	1	.01000	.01000	0	.01000	.01000	0	.0100
2	10	.1709	.0921	−46	.1273	.0970	−24	.100
	5	.1618	.0931	−42	.1222	.0975	−20	.100
	2	.1360	.0960	−29	.1111	.0988	−11	.100
	1	.1000	.1000	0	.1000	.1000	0	.100

*When $k=2$, $E_p(t_k)=V_p(B_k)$, and R.B. $=0$ for all p.

In comparing B_k, $k=0$, 1, 2, for fixed p, notice in table 4 that the differences among the three estimators are less pronounced when N is smaller. Furthermore, table 4 illustrates the way in which the choice of spacing for x-values affects the efficiency of the estimators B_k, and t_k for fixed p and k.

Thus, it is seen in table 4 that for given p and k, $V_p(B_k)$ is smallest when $N=1$. This result is not unexpected since it is well known for $p=0$ that the best estimate of the slope of a line through the origin is obtained with all observations made at one point as far away from the origin as possible. The same is true for any $p\leq 2$, as follows. When all x_i are equal to the largest, x_n, then from eq (2–1)

$$V_p(B_k)=\sigma^2/n\,x_n^{2-p}.$$

This choice of the x_i minimizes $V_p(B_k)$, since

$$\left(\sum_{i=1}^{n} x_i^{2-k}\right)^2\leq\left(\sum_{i=1}^{n} x_i^{2-2k+p}\right)\left(\sum_{i=1}^{n} x_i^{2-p}\right)$$

and

$$\sum_{i=1}^{n} x_i^{2-p}\leq n\,x_n^{2-p} \text{ if } p\leq 2.$$

Minimizing the variance of the estimate of B_k by choosing $N=1$ is desirable only when the experimenter does not need observations at intermediate points to check on linearity. The dependence of the variance of y_i on x_i is irrelevant, of course, when all x_i are equal; for then all observations y_i have equal variances.

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Existence of k-Edge Connected Ordinary Graphs With Prescribed Degrees[1]

Jack Edmonds

(February 25, 1964)

An *ordinary graph* G is a set of objects called *nodes* and a family of unordered pairs of the nodes called *edges*. The *degree* of a node in G is the number of edges in G which contain it. G is called *connected* if it is not the union of two disjoint nonempty subgraphs. A graph H is called k-edge connected if deleting any fewer than k edges from H leaves a connected graph. It is proved that there exists a k-edge connected graph H for $k > 1$ with prescribed integer degrees d_i if and only if there exists an ordinary graph with these degrees and all $d_i \geq k$. There exists a 1-connected (i.e., connected) ordinary graph with prescribed positive integer degrees d_i if and only if there exists an ordinary graph with these degrees and $\sum_{i=1}^{n} d_i \geq 2(n-1)$.

An *ordinary graph* G is a finite set of objects and a family of two-member subsets of the objects. The objects are called the *nodes* of G and the pairs are called the *edges* of G. An edge and a node are said to *meet* if one contains the other. The *degree* of a node in G is the number of edges in G which it meets.

A *cut* of graph G, denoted by $(S,\bar{S})$, is a partition of the nodes of G into two nonempty subsets S and $\bar{S}$. The union of S and $\bar{S}$ is all the nodes of G. The intersection of S and $\bar{S}$ is empty. An edge e *in* cut $(S,\bar{S})$ of G is an edge of G which meets one node in S and one node in $\bar{S}$. The *value* of cut $(S,\bar{S})$ is the number of edges in it.

Graph G is called k-edge connected if it has no cut with value less than k. The usual meaning of *connected* for graphs is the same as the meaning of 1-edge connected. An equivalent definition of k-edge connected for G is that G cannot be made nonconnected by deleting fewer than k edges from G.

Hoffman, Fulkerson, and McAndrew[2] describe necessary and sufficient conditions for a set of integers to be the degrees of the nodes of some ordinary graph, and give references to related work. Here, we show that simple additional conditions are necessary and sufficient for the set of integers to be the degrees of the nodes of some k-edge connected ordinary graph (prescribed k).

THEOREM. *Integers* d_i, $i = 1, \ldots, n$, $(n > 1)$, *are the respective degrees of the nodes* v_i *of some* k*-edge connected ordinary graph if and only if*

(1) *they are the degrees of the nodes of some ordinary graph,*
(2) *all* $d_i \geq k$,
(3) *where* $k = 1$, $\sum_{i=1}^{n} d_i \geq 2(n-1)$.

The "only if" part of the theorem is almost obvious. Certainly condition (1) is necessary. Condition (2) is necessary since a cut of a graph G which puts one node, say v, in one part and the other nodes of G in the other part has cut value equal to the degree of v.

For $k = 1$, condition (2)—that the d_i's be positive—is necessary but not sufficient. Let G be a connected graph with n nodes. There must be an edge, say e_1, joining node v_1 of G to one of the other nodes of G, say v_2. Recursively, there must be an edge, say e_i, joining one of the nodes $v_1, \ldots, v_i$ $(1 \leq i \leq n-1)$ of G to one of the other nodes of G, say v_{i+1}. Clearly each of the edges $e_i (i = 1, \ldots, n-1)$ is distinct. Therefore the number of edges in G is not less than $n-1$. Let d_i be the degree of node $v_i (i = 1, \ldots, n)$ in G. Because each edge of G contributes twice to a degree in G, $\sum d_i$ equals twice the number of edges in G. Therefore, $\sum d_i \geq 2(n-1)$.

A (simple) *path joining nodes* a *and* b is a connected graph which has degree $= 1$ at its nodes a and b and degree $= 2$ at its other nodes. The edges $e_i (i = 1, \ldots, n-1)$ described in the last paragraph form with their nodes a tree. A *tree* is a graph which contains exactly one path joining each pair of its nodes. A tree contains one less edge than nodes. A graph G contains a tree containing all the nodes of G if and only if G is connected. The edges and nodes of any graph partition uniquely into one or more connected *component* graphs. Two nodes in a graph G are joined by a path in G if and only if they are in the same component of G.

Suppose there exists an ordinary graph H, not necessarily connected, with positive integer degrees $d_i (i = 1, \ldots, n)$ at its nodes. Suppose $\sum d_i \geq 2(n-1)$ and hence H has at least $n-1$ edges. If H is not connected, i.e., has more than one component, we shall see that another ordinary graph H' can be constructed from H which has the same degrees at all the nodes and which has one less component than H.

[1] Supported by the Army Research Office (Durham).
[2] D. R. Fulkerson, A. J. Hoffman, and M. H. McAndrew, Some properties of graphs with multiple edges, to appear in the Canadian Journal of Mathematics.

Thus it will follow by induction that there exists a connected ordinary graph with these degrees.

Suppose H has components $H_1, \ldots, H_p$ with $n_1, \ldots, n_p$ nodes respectively. Each $H_j (j = 1, \ldots, p)$ contains a tree T_j with $n_j - 1$ edges and all n_j nodes of H_j. Since

$$\sum_{j=1}^{p} n_j = n,$$

we have

$$\sum_{j=1}^{p} (n_j - 1) < n - 1 \text{ for } p > 1.$$

Since there are at least $n-1$ edges in H, there must be an edge e of H not in any of the trees T_j. However e must have both of its nodes, say a and b, in one of these trees, say T_1.

Since component H_2 contains a node and all d_i's are positive, H_2 contains an edge, say e_2, which meets nodes, say a_2 and b_2, in H_2. Form a new graph H' from H by deleting from H edges e and e_2 and then adjoining a new edge e_a joining a and a_2 and a new edge e_b joining b and b_2. Because a_2 and b_2 are in a different component of H than a and b, there are no edges in H already joining a_2 to a and b_2 to b. Hence H' is a validly constructed ordinary graph with the same degrees at its nodes as in H. It is easy to verify that H_1 and H_2 are hereby replaced by a single component in H'. Therefore H' has one less component than H. This completes the proof of the theorem for the case $k = 1$.

The proof for the case $k > 1$ is more tricky. Here we don't have an analog of trees or a straightforward decomposition into k-edge connected components to help us.

For prescribed integer $k > 1$, assume that all $d_i \geq k (i = 1, \ldots, n)$ and assume there exists an ordinary graph whose nodes v_i have degrees d_i. It follows that there is a connected ordinary graph H whose nodes v_i have degrees d_i, since $d_i \geq 2$ implies

$$\sum d_i \geq 2n > 2(n-1).$$

Assuming that H is only h-edge connected where $1 \leq h < k$, we shall show how to construct a new ordinary graph H' whose nodes v_i have degrees d_i, and which is at least h-edge connected, and which has fewer cuts of value h than H does. From this it will follow by induction that there is a k-edge connected ordinary graph G whose nodes v_i have degrees d_i.

Clearly H has h-valued cuts $(A, \overline{A})$ and $(B, \overline{B})$ such that sets A and B are disjoint and such that neither A nor B contain a proper subset S for which $(S, \overline{S})$ is only an h-valued cut. It may or may not be true that $A = \overline{B}$ and $\overline{A} = B$. In any case, because H is connected, there must be a path P joining a node, a_1, in A to a node, b_1, in B where P has no other nodes in A or B.

Because node a_1 has degree at least $k > h$, and because there are only h edges with one node in A and one node not in A, node a_1 must meet an edge e_a whose other node a_2 is in A and meets no edge with a node not in A. Similarly, node b_1 must meet an edge e_b whose other node b_2 is in B and meets no edge with a node not in B.

Construct a new ordinary graph H' by deleting edges e_a and e_b and then adjoining a new edge e'_a joining a_1 to b_2 and a new edge e'_b joining b_1 to a_2. Graph H' has the same degrees at the various nodes as H. The value of the cuts $(A, \overline{A})$ and $(B, \overline{B})$ is increased to $h+2$.

It remains to show that replacing H by H' introduces no new cuts of value $\leq h$. Any cut, say $(D, \overline{D})$, whose value decreases in going from H to H' must have a_1 and b_2 together in one of the parts, say D, and a_2 and b_1 together in the other part, $\overline{D}$. Suppose cut $(D, \overline{D})$ in H' has value h or less.

Consider the mutually disjoint sets of nodes $A \cap D$, $A \cap \overline{D}$, $B \cap D$, $B \cap \overline{D}$. Path P, defined with respect to H, is also entirely in H' and it joins node a_1 in $A \cap D$ to node b_1 in $B \cap \overline{D}$. Hence some edge e_0 of P is an edge of the cut $(D, \overline{D})$ of H'. Since P contains no nodes of $A \cap \overline{D}$ or $B \cap D$, edge e_0 neither meets one node in $A \cap \overline{D}$ and one node in $A \cap D$ nor meets one node in $B \cap D$ and one node in $B \cap \overline{D}$.

Partition the edges of the cut $(D, \overline{D})$ in H' into three classes E_A, E_B, E_C: E_A consisting of all edges connecting $A \cap D$ with $A \cap \overline{D}$, E_B consisting of all edges connecting $B \cap D$ with $B \cap \overline{D}$, E_C being formed by the remaining edges. As has been seen above, E_C contains e_0; hence the cardinality of E_C is at least one. Since the cardinalities of E_A, E_B, E_C add up to at most h, either the cardinality of E_A or the cardinality of E_B must be bounded by [3] $[(h-1)/2]$. By symmetry, suppose the former is true. Then we have, counting e_a, at most $[(h-1)/2]+1$ edges in H with a node in $A \cap D$ and a node in $A \cap \overline{D}$.

Since the value of cut $(A, \overline{A})$ in H is h, either $A \cap D$ or $A \cap \overline{D}$ contains nodes of no more than $[h/2]$ edges of cut $(A, \overline{A})$. Every edge of H with precisely one node in $A \cap D$ (in $A \cap \overline{D}$) has its other node in $A \cap \overline{D}$ (in $A \cap D$) or in $\overline{A}$. Therefore either cut $(A \cap D, \overline{A \cap D})$ or cut $(A \cap \overline{D}, \overline{A \cap \overline{D}})$ of H has value at most

$$[h/2] + [(h-1)/2] + 1 = h.$$

However this contradicts the fact that A contains no proper subset S such that cut $(S, \overline{S})$ in H has value at most h. Therefore, our assumption that cut $(D, \overline{D})$ in H' has value at most h is incorrect. Therefore replacing H by H' introduces no new cuts of value at most h. The theorem is proved.

Mr. Alvin Owen (in conversation) has posed an interesting related problem. What are necessary and sufficient conditions for a set of integers

$$\{d_i\} (i = 1, \ldots, n)$$

so that *all* ordinary graphs whose nodes have degrees precisely $\{d_i\}$ are connected? Or, looking for the negative of these conditions, what are necessary and sufficient conditions for $\{d_i\}$ to be the degrees of the nodes of some nonconnected ordinary graph?

Note added in proof: D. R. Fulkerson has called to my attention a related paper by himself and L. S. Shapley, Minimal k-arc connected graphs, RAND Report P-2371 (1961). Their main result states that the fewest edges required in a k-edge connected ordinary graph with n nodes is $[kn-1)/2]$.

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[3] Here $[x]$ means "the greatest integer not greater than x."

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Inequalities for Solutions of Mixed Boundary Value Problems for Elastic Plates

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In this paper a number of explicit a priori inequalities for solutions of the plate equation are derived. These inequalities together with the Rayleigh-Ritz technique may be used to compute error bounds in various mixed boundary value problems for elastic plates.

1. Introduction

In two recent papers [2,3][1] the authors presented methods for obtaining pointwise bounds in the three most common boundary value problems for elastic plates. These bounds were of a priori type, that is they held for a class of functions required to satisfy only smoothness conditions. Hence one could approximate the (unknown) solution of one of these problems in terms of essentially arbitrary functions, and the inequalities gave bounds on the error.

In this paper we derive similar a priori bounds in the three most common mixed boundary value problems for elastic plates. For simplicity we consider only the case of a simply connected region R whose boundary Σ consists of two disjoint portions Σ_1, and Σ_2 (each connected) on which different sets of boundary conditions are imposed. It will be clear how the results are to be extended if Σ_1, and/or Σ_2 are not connected or if R is multiply connected.

In this paper we shall restrict our attention to the problem of obtaining bounds for the L_2 integrals of an arbitrary sufficiently smooth function w in terms of L_2 integrals of quantities which are data whenever the arbitrary function w is actually the solution u to the problem in question. By use of mean value inequalities and the Rayleigh-Ritz technique as indicated in [2,3], the desired pointwise bounds are then obtained. The well known Rayleigh-Ritz technique consists in choosing $w = u - \sum_{i=1}^{N} a_i \Phi_i$, where the Φ_i are N linearly independent sufficiently smooth functions, and the a_i are determined in such a way as to minimize the terms involving the data of u.

The particular problems treated here are the following:

PROBLEM I:

$$\Delta^2 u \quad \text{prescribed in } R$$

$$u, \frac{\partial u}{\partial n} \quad \text{given on } \Sigma_1$$

$$u, M(u) \quad \text{given on } \Sigma_2$$

PROBLEM II:

$$\Delta^2 u \quad \text{prescribed in } R$$

$$u, \frac{\partial u}{\partial n} \quad \text{given on } \Sigma_1$$

$$M(u), V(u) \quad \text{given on } \Sigma_2$$

[1] Figures in brackets indicate the literature references at the end of this paper.

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PROBLEM III: $\Delta^2 u$ prescribed in R

$u, M(u)$ given on Σ_1

$M(u), V(u)$ given on Σ_2.

Here Δ^2 is the biharmonic operator $\frac{\partial^4}{\partial x^4}+2\frac{\partial^4}{\partial x^2\partial y^2}+\frac{\partial^4}{\partial y^4}$ (Δ denotes the Laplace operator), and $\frac{\partial}{\partial n}$ denotes the normal derivative directed outward on Σ. Also

$$M(u)=\Delta u-(1-\sigma)\left(\frac{\partial^2 u}{\partial s^2}+\frac{1}{\rho}\frac{\partial u}{\partial n}\right) \tag{1.1}$$

and

$$V(u)=\frac{\partial}{\partial n}(\Delta u)+(1-\sigma)\left[\frac{\partial^3 u}{\partial s^2\partial n}-\frac{\partial}{\partial s}\left(\frac{1}{\rho}\frac{\partial u}{\partial s}\right)\right] \tag{1.2}$$

where $\frac{\partial}{\partial s}$ denotes the derivative with respect to arc length on Σ, σ is Poisson's ratio and ρ is the radius of curvature. The quantities $M(u)$ and $V(u)$ are proportional to the normal moment and the reaction normal to the plate on Σ, respectively.

In subsequent sections we shall refer to the bounding of integrals of an arbitrary function w in terms of L_2 integrals of its "data." For example, in problem I the data of w are the quantities $\Delta^2 w$ in R, w and $\frac{\partial w}{\partial n}$ on Σ_1 and w and $M(w)$ on Σ_2. In the other two problems the "data" of w are defined analogously.

Throughout this paper we assume the boundary Σ of R to be sufficiently smooth so that all of the applications of the divergence theorem used are valid. It will be apparent that if the derivatives of the curvature of Σ are continuous then there is no question of the validity of the indicated operations provided the functions themselves are smooth enough. It will also be clear that less smoothness of Σ can actually be tolerated in each problem, but the determination of the minimum smoothness requirements is not investigated in this paper.

We make use of the summation convention throughout and employ a comma to denote differentiation, e.g., $w_{,i}=\frac{\partial w}{\partial x^i}$. We denote the points at which Σ_1 and Σ_2 join as P_1 and P_2. Arc length will be measured along Σ in the counterclockwise direction.

Additional notation will be defined as the need arises in the text. We turn now to the question of obtaining bounds in problems I, II, and III. These will be discussed separately in sections 2, 3, and 4. Certain auxiliary inequalities which will be required in treating the three mixed problems will be derived in appendix A.

2. Problem I

In a recent paper [4, eq (2.1)] the authors derived the following inequality for a function w with piecewise continuous second derivatives:

$$\left\{\int_R w^2 dA\right\}^{1/4} \leq K_1\left\{\oint_\Sigma w^2 dS\right\}^{1/4}+K_2\left\{\int_R (\Delta w)^2 dA\right\}^{1/4}, \tag{2.1}$$

where the constants K_1 and K_2 were explicit. (As indicated in [4,5] appropriate constants K_1 and K_2 are obtained by making use of results of Payne and Weinberger [10]). Using (2.1) as a starting point we shall derive the desired L_2 inequality in problem I.

We first define $E(w,w)$ as

$$E(w, w)=\int_R [\sigma(\Delta w)^2+(1-\sigma)w_{,ij}w_{,ij}]dA. \tag{2.2}$$

It is well known that $E(u, u)$ is proportional to the strain energy. Since $(\Delta w)^2 \leq 2w_{,ij}w_{,ij}$ it follows that

$$\int_R (\Delta w)^2 dA \leq \frac{2}{1+\sigma} E(w, w). \tag{2.3}$$

Thus inserting (2.3) into (2.1) we obtain

$$\left\{\int_R w^2 dA\right\}^{1/2} \leq K_1\left\{\oint_\Sigma w^2 dS\right\}^{1/2} + \tilde{K}_2\left\{E(w, w)\right\}^{1/2} \tag{2.4}$$

where

$$\tilde{K}_2 = K_2[2(1+\sigma)^{-1}]^{1/2}.$$

We seek now to bound $E(w, w)$ in terms of the data of problem I. In order to simplify our computations we shall assume that

$$\lim_{Q \to P_1, P_2} \frac{\partial w(Q)}{\partial n} = 0, \quad \lim_{Q \to P_1, P_2} \frac{\partial^2 w(Q)}{\partial s \partial n} = 0 \tag{2.5}$$

where Q is a point on Σ_1. This restriction is by no means necessary, since bounds could be derived which involve the values of $\frac{\partial w}{\partial n}$ and $\frac{\partial^2 w}{\partial s \partial n}$ at P_1 and P_2. In applying the a priori type inequalities, however, w will be set equal to $u - \Phi$ where u is the solution to problem I and Φ is an approximating function. The function Φ will then be represented as a linear combination of N linearly independent functions Φ_i with arbitrary real coefficients. The restriction (2.5) prescribes four relations among the coefficients. The Rayleigh-Ritz technique together with these four conditions then determine the coefficients. Hence, for practical purposes, (2.5) is no restriction at all. Condition (2.5) permits us to treat the cases in which Σ_1 and Σ_2 do not have end points in common (e.g., certain multiply connected configurations) and the case in which points P_1 and P_2 do occur, in a unified manner.

In order to bound $E(w, w)$ in terms of the data of problem I, we decompose w as follows:

$$w = B + \varphi \tag{2.6}$$

where

$$\begin{gathered} \Delta^2 B = 0 \text{ in } R \\ B = w \text{ on } \Sigma \\ \frac{\partial B}{\partial n} = \begin{cases} \frac{\partial w}{\partial n} \text{ on } \Sigma_1 \\ 0 \text{ on } \Sigma_2. \end{cases} \end{gathered} \tag{2.7}$$

Clearly then

$$\begin{gathered} \Delta^2 \varphi = \Delta^2 w \text{ in } R \\ \varphi = 0 \text{ on } \Sigma \\ \frac{\partial \varphi}{\partial n} = 0 \text{ on } \Sigma_1 \\ M(\varphi) = M(w) - M(B) \text{ on } \Sigma_2. \end{gathered} \tag{2.8}$$

We now consider the identity

$$E(w,w)=E(B,B)+2E(w,\varphi)-E(\varphi,\varphi) \tag{2.9}$$

and note that

$$E(w,\varphi)=\int_{\Sigma_2}\frac{\partial\varphi}{\partial n}M(w)dS+\int_R\varphi\Delta^2 w dA\cdot \tag{2.10}$$

By Schwarz's inequality we have

$$|E(w,\varphi)|\leqslant\left\{\int_{\Sigma_2}\left(\frac{\partial\varphi}{\partial n}\right)^2dS\int_{\Sigma_2}M^2(w)dS\right\}^{1/2}+\left\{\int_R\varphi^2dA\int_R(\Delta^2w)^2dA\right\}^{1/2}\cdot \tag{2.11}$$

Since φ vanishes on Σ, it follows from the duality principle of Fichera [8] that

$$\left\{\int_\Sigma\left(\frac{\partial\varphi}{\partial n}\right)^2dS\right\}^{1/2}\leqslant K_1\left\{\int_R(\Delta\varphi)^2dA\right\}^{1/2} \tag{2.12}$$

where K_1 is the same constant as the K_1 appearing in (2.1). (In the determination of the explicit value for K_1 in [5] and [10] the quantity was actually determined for (2.12) rather than (2.1).) It is well known also that if φ has piecewise continuous second derivatives in R and vanishes on Σ then

$$\left\{\int_R\varphi^2dA\right\}^{1/2}\leqslant\frac{1}{\lambda_1}\left\{\int_R(\Delta\varphi)^2dA\right\}^{1/2} \tag{2.13}$$

where λ_1 is the first eigenvalue in the fixed membrane problem for R. If this number is not known, lower bounds for it can be easily computed from monotony principles, the Faber-Krahn [7, 9] inequality, or other considerations. Inserting (2.12) and (2.13) into (2.11) we obtain

$$\begin{aligned}|E(w,\varphi)|&\leqslant\left[K_1\left\{\int_{\Sigma_2}[M(w)]^2dS\right\}^{1/2}+\frac{1}{\lambda_1}\left\{\int_R(\Delta^2w)^2dA\right\}^{1/2}\right]\left\{\int_R(\Delta\varphi)^2dA\right\}^{1/2}\\&\leqslant\left[K_1\left\{\int_{\Sigma_2}[M(w)]^2dS\right\}^{1/2}+\frac{1}{\lambda_1}\left\{\int_R(\Delta^2w)^2dA\right\}^{1/2}\right]\left[\frac{2}{1+\sigma}E(\varphi,\varphi)\right]^{1/2}\end{aligned} \tag{2.14}$$

the last inequality resulting from (2.3). From (2.14) and (2.9) it follows then that

$$E(w,w)\leqslant E(B,B)+2\left(\frac{2}{1+\sigma}\right)^{1/2}\left[K_1\left\{\int_{\Sigma_2}[M(w)]^2dS\right\}^{1/2}+\frac{1}{\lambda_1}\left\{\int_R(\Delta^2w)^2dA\right\}^{1/2}\right][E(\varphi,\varphi)]^{1/2}-E(\varphi,\varphi) \tag{2.15}$$

or, using the arithmetic geometric-mean inequality,

$$E(w,w)\leqslant E(B,B)+\frac{2}{1+\sigma}\left[K_1\left\{\int_{\Sigma_2}[M(w)]^2dS\right\}^{1/2}+\frac{1}{\lambda_1}\left\{\int_R(\Delta^2w)^2dA\right\}^{1/2}\right]^2. \tag{2.16}$$

By combining (2.4) and (2.16) we now have a bound for $\int_R w^2dA$ in terms of L_2 integrals of the data and $E(B,B)$, where B is the solution to (2.7). We seek now to bound $E(B,B)$ in terms of the data of w.

From the divergence theorem we have

$$\begin{aligned}E(B,B)&=\int_R(\Delta B)^2dA+(1-\sigma)\int_R[B,_{ij}B,_{ij}-(\Delta B)^2]dA\\&=\int_R(\Delta B)^2dA+(1-\sigma)\oint_\Sigma B,_i\left(n_j\frac{\partial}{\partial x^i}-n_i\frac{\partial}{\partial x^j}\right)B,_jdS\\&=\int_R(\Delta B)^2dA+(1-\sigma)\oint_\Sigma\left\{\frac{\partial B}{\partial s}\frac{\partial^2B}{\partial s\partial n}-\frac{\partial B}{\partial n}\frac{\partial^2B}{\partial s^2}-\frac{1}{\rho}\left[\left(\frac{\partial B}{\partial n}\right)^2+\left(\frac{\partial B}{\partial s}\right)^2\right]\right\}dS.\end{aligned} \tag{2.17}$$

Now if $\frac{\partial B}{\partial s}$ and $\frac{\partial B}{\partial n}$ are continuous on Σ and either $\frac{\partial^2 B}{\partial s \partial n}$ $\left(\text{or } \frac{\partial^2 B}{\partial s^2}\right)$ is square integrable on Σ, then we may integrate the first (or second) term in the boundary integral on the right by parts on Σ to obtain, for instance,

$$E(B,B)=\int_R (\Delta B)^2 dA - 2(1-\sigma)\int_{\Sigma_1} \frac{\partial w}{\partial n}\frac{\partial^2 w}{\partial s^2} - \oint_\Sigma \frac{1}{\rho}\left(\frac{\partial w}{\partial s}\right)^2 dS - \int_{\Sigma_1} \frac{1}{\rho}\left(\frac{\partial w}{\partial n}\right)^2 dS. \tag{2.18}$$

We have also made use of the boundary conditions (2.7). An application of the arithmetic-geometric mean inequality yields for any positive $\hat{\alpha}$

$$E(B,B) \leq \int_R (\Delta B)^2 dA + (1-\sigma)^2 \oint_\Sigma \hat{\alpha}\left(\frac{\partial^2 w}{\partial s^2}\right) dS + \int_{\Sigma_1} (\hat{\alpha}^{-1}-\rho^{-1})\left(\frac{\partial w}{\partial n}\right)^2 dS - \oint_\Sigma \rho^{-1}\left(\frac{\partial w}{\partial s}\right)^2 dS. \tag{2.19}$$

An inequality bounding the first term on the right of (2.19) in terms of the data of w is derived in appendix A. The insertion of (A.25) into (2.19) together with (2.7) results in the inequality

$$E(B,B) \leq \hat{d}_1 \oint_\Sigma w^2 dS + \hat{d}_2 \oint_\Sigma \left(\frac{\partial w}{\partial s}\right)^2 dS + \hat{d}_3 \int_{\Sigma_1} \left(\frac{\partial w}{\partial n}\right)^2 dS + \hat{d}_4 \oint_\Sigma \left(\frac{\partial^2 w}{\partial s^2}\right)^2 dS + \hat{d}_5 \int_{\Sigma_1} \left(\frac{\partial^2 w}{\partial s \partial n}\right)^2 dS, \tag{2.20}$$

with explicit constants $\hat{d}_i$. If the region R is simply connected we can actually obtain the inequality (see appendix A, in particular (A.26)–(A.28))

$$E(B,B) \leq \hat{d}_6 \oint_\Sigma \frac{\partial B_{,i}}{\partial s}\frac{\partial B_{,i}}{\partial s} dS. \tag{2.21}$$

We now insert (2.20) and (2.16) back into (2.4) to obtain the desired inequality, which may be written in the form

$$\begin{aligned}\int_R w^2 dA \leq A_1 \oint_\Sigma w^2 dS + A_2 \oint_\Sigma \left(\frac{\partial w}{\partial s}\right)^2 dS + A_3 \oint_\Sigma \left(\frac{\partial^2 w}{\partial s^2}\right)^2 dS \\ + A_4 \int_{\Sigma_1} \left(\frac{\partial w}{\partial n}\right)^2 dS + A_5 \int_{\Sigma_1} \left(\frac{\partial^2 w}{\partial s \partial n}\right)^2 dS + A_6 \int_{\Sigma_2} [M(w)]^2 dS + A_7 \int_R (\Delta^2 w)^2 dA.\end{aligned} \tag{2.22}$$

Here again the constants A_i may be explicitly computed.

In many of our intermediate inequalities sharper results could have been obtained by leaving certain positive weight functions under the integral signs rather than factoring them out.

It is now possible to employ the technique used in [2] and [3] to obtain pointwise bounds in R for w and its derivatives in terms of L_2 integrals of the data of problem I. It appears likely that by using the results of Bramble and Hubbard [1] it would be possible, if sufficient smoothness of the boundary and data is assumed, to obtain an inequality of the form (2.22) in which A_2, A_3, and A_5 are zero. However, the remaining constants would then be more complicated functions of the geometry of Σ_1 and Σ_2.

3. Problem II

We assume in this section that R and the portion Σ_2 of Σ are such that we can define a vector field f^i which has piecewise continuous first derivatives in R and satisfies

$$\begin{aligned} f^i_{,i} - f^i f^i > p > 0 \text{ in } R \\ -f^i n_i > q > 0 \text{ on } \Sigma_2. \end{aligned} \tag{3.1}$$

This is of course always possible if Σ is sufficiently smooth and Σ_1 is nonempty. In fact, if we take any sufficiently smooth function φ which is superharmonic in R and whose normal derivative is positive on Σ_2 then one such f^i is given by

$$f^i = \frac{-\varphi_{,i}}{\varphi + a}$$

where a is chosen so large that $\varphi + a > 0$ in R.

In this section we derive a bound for the L_2 integral of an arbitrary C^3 function w in terms of L_2 integrals of the data of problem II. Again in the application of our inequalities we shall set $w = u - \Phi$, where u is the solution to problem II and Φ is the approximating function. In this section we assume that Φ has been so chosen that

$$\lim_{Q \to P_1, P_2} w(Q) = \lim_{Q \to P_1, P_2} \frac{\partial w(Q)}{\partial s} = \lim_{Q \to P_1, P_2} \frac{\partial^2 w(Q)}{\partial s^2} = \lim_{Q \to P_1, P_2} \frac{\partial w(Q)}{\partial n} = \lim_{Q \to P_1, P_2} \frac{\partial^2 w(Q)}{\partial s \partial n} = 0. \tag{3.2}$$

If we choose

$$\Phi = \sum_{i=1}^{N} a_i \Phi_i \tag{3.3}$$

where the Φ_i are linearly independent functions, then (3.2) imposes ten conditions on the a_i. If Σ_1 and Σ_2 have no points in common then conditions (3.2) are unnecessary.

We now consider the following application of the divergence theorem.

$$\oint_{\Sigma} f^i n_i w^2 dS = \int_R f^i_{,i} w^2 dA + 2 \int_R f^i w w_{,i} dA. \tag{3.4}$$

Upon rearranging and applying the arithmetic-geometric mean inequality, we obtain

$$\int_R (f^i_{,i} - f^i f^i) w^2 dA \leq \oint_{\Sigma} f^i n_i w^2 dS + \int_R |\text{grad } w|^2 dA. \tag{3.5}$$

Using (3.1) we have

$$p \int_R w^2 dA + q \int_{\Sigma_2} w^2 dS \leq \int_{\Sigma_1} f^i n_i w^2 dS + \int_R |\text{grad } w|^2 dA. \tag{3.6}$$

Let us now rewrite (3.4) with w^2 replaced by $|\text{grad } w|^2$. It follows then in the same way that

$$p \int_R |\text{grad } w|^2 dA + q \int_{\Sigma_2} |\text{grad } w|^2 dS \leq \int_{\Sigma_1} f^i n_i |\text{grad } w|^2 dS + \int_R w_{,i} w_{,i} dA. \tag{3.7}$$

Combining the two inequalities and dropping the integrals over Σ_2, we obtain the inequality

$$\int_R w^2 dA \leq p^{-1} \int_{\Sigma_1} f^i n_i w^2 dS + p^{-2} \int_{\Sigma_1} f^i n_i |\text{grad } w|^2 dS + p^{-2} \int_R w_{,i} w_{,i} dA. \tag{3.8}$$

Clearly for $\sigma \geq 0$

$$\int_R w_{,i} w_{,i} dA \leq (1 - \sigma)^{-1} E(w, w). \tag{3.9}$$

If $\sigma < 0$, the factor $(1-\sigma)^{-1}$ must be replaced by $2(2-\sigma)^{-1}$. We assume, however, in the following that $\sigma \geq 0$. Then

$$\int_R w^2 dA \leq p^{-1}\int_{\Sigma_1} f^i n_i w^2 dS + p^{-2}\int_{\Sigma_1} f^i n_i |\text{grad } w|^2 dS + [p^2(1-\sigma)]^{-1} E(w, w). \tag{3.10}$$

In order to bound $E(w, w)$ in terms of the desired quantities, we decompose w as follows:

$$w = B + \varphi \tag{3.11}$$

where B satisfies

$$\begin{gathered}\Delta^2 B = 0 \text{ in } R \\ B = \begin{cases} w \text{ on } \Sigma_1 \\ 0 \text{ on } \Sigma_2 \end{cases} \\ \frac{\partial B}{\partial n} = \begin{cases} \dfrac{\partial w}{\partial n} \text{ on } \Sigma_1 \\ 0 \text{ on } \Sigma_2 \end{cases}\end{gathered} \tag{3.12}$$

The function φ then satisfies

$$\begin{gathered}\Delta^2 \varphi = \Delta^2 w \text{ in } R \\ \varphi = \frac{\partial \varphi}{\partial n} = 0 \text{ on } \Sigma_1 \\ M(\varphi) = M(w) - M(B),\ V(\varphi) = V(w) - V(B) \text{ on } \Sigma_2 .\end{gathered} \tag{3.13}$$

We again make use of the identity

$$E(w, w) = E(B, B) + 2E(\varphi, w) - E(\varphi, \varphi) \tag{3.14}$$

where now

$$E(\varphi, w) = \int_{\Sigma_2} \frac{\partial \varphi}{\partial n} M(w) dS - \int_{\Sigma_2} \varphi V(w) dS + \int_R \varphi \Delta^2 w dA. \tag{3.15}$$

We apply the Schwarz inequality as follows:

$$\begin{aligned}\left|E(\varphi, w)\right| \leq \left\{q\int_{\Sigma_2}\left(\frac{\partial \varphi}{\partial n}\right)^2 dS + p^2\int_R \varphi^2 dA + pq\int_{\Sigma_2}\varphi^2 dS\right\}^{1/2} \Bigg\{q^{-1}\int_{\Sigma_2}[M(w)]^2 dS \\ + p^{-2}\int_R (\Delta^2 w)^2 dA + p^{-1}q^{-1}\int_{\Sigma_2}(V(w))^2 dS\Bigg\}^{1/2}.\end{aligned} \tag{3.16}$$

Now, in view of (3.6), the first term on the right of (3.16) may be bounded by

$$q\int_{\Sigma_2}\left(\frac{\partial \varphi}{\partial n}\right)^2 dS + p^2\int_R \varphi^2 dA + pq\int_{\Sigma_2}\varphi^2 ds \leq p\int_R |\text{grad } \varphi|^2 dA + q\int_{\Sigma_2}|\text{grad } \varphi|^2 dS. \tag{3.17}$$

We have also made use of the fact that $\left(\frac{\partial \varphi}{\partial n}\right)^2 \leq |\text{grad } \varphi|^2$. But from (3.7) it follows then that

$$p\int_R |\text{grad } \varphi|^2 dA + q\int_{\Sigma_2}|\text{grad } \varphi|^2 dS \leq \int_R \varphi_{,ij}\varphi_{,ij} dA = (1-\sigma)^{-1}E(\varphi, \varphi). \tag{3.18}$$

Thus

$$|E(\varphi, w)| \leq \{(1-\sigma)^{-1}E(\varphi, \varphi)\}^{1/2}\left\{q^{-1}\int_{\Sigma_2}(M(w))^2 dS + p^{-2}\int_R (\Delta^2 w)^2 dA + p^{-1}q^{-1}\int_{\Sigma_2}(V(w))^2 dS\right\}^{1/2}. \tag{3.19}$$

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The insertion of (3.19) into (3.14), followed by an application of the arithmetic-geometric mean inequality, yields

$$E(w, w) \leq E(B, B) + (1-\sigma)^{-1}\left\{q^{-1}\int_{\Sigma_2}(M(w))^2 dS + p^{-2}\int_R (\Delta^2 w)^2 dA + p^{-1}q^{-1}\int_{\Sigma_2}(V(w))^2 dS\right\}. \tag{3.20}$$

A combination of (3.10) and (3.20) thus gives us a bound for $\int_R w^2 dA$ in terms of the data of w and $E(B,B)$. We seek now a bound for $E(B,B)$ in terms of the data of w.

From (2.17), it follows as before that

$$E(B, B) = \int_R (\Delta B)^2 dA + 2(1-\sigma)\int_{\Sigma_1}\frac{\partial w}{\partial s}\frac{\partial^2 w}{\partial s \partial n}\, dS - \int_{\Sigma_1}\frac{1}{\rho}\left[\left(\frac{\partial w}{\partial s}\right)^2 + \left(\frac{\partial w}{\partial n}\right)^2\right] dS. \tag{3.21}$$

Using the arithmetic-geometric mean inequality we obtain for positive α_8

$$E(B, B) \leq \int_R (\Delta B)^2 dA + (1-\sigma)^2\alpha_8\int_{\Sigma_1}\left(\frac{\partial^2 w}{\partial s \partial n}\right)^2 dS + \int_{\Sigma_1}(\alpha_8^{-1} - \rho^{-1})\left(\frac{\partial w}{\partial s}\right)^2 dS - \int_{\Sigma_1}\rho^{-1}\left(\frac{\partial w}{\partial n}\right)^2 dS. \tag{3.22}$$

As in the previous section we require a bound for $\int_R (\Delta B)^2 dA$ in terms of the data of w. The desired bound was computed in appendix A and is given by (A.25), i.e.,

$$\int_R (\Delta B)^2 dA \leq \overline{d}_1\oint_\Sigma B^2 dS + \overline{d}_2\oint_\Sigma\left(\frac{\partial B}{\partial s}\right)^2 dS + \overline{d}_3\oint_\Sigma\left(\frac{\partial B}{\partial n}\right)^2 dS + \overline{d}_4\oint_\Sigma\left(\frac{\partial^2 B}{\partial s \partial n}\right)^2 dS + \overline{d}_5\oint_\Sigma\left(\frac{\partial^2 B}{\partial s^2}\right)^2 dS. \tag{3.23}$$

This inequality may be rewritten, using (3.12), as

$$\int_R (\Delta B)^2 dA \leq \overline{d}_1\int_{\Sigma_1} w^2 dS + \overline{d}_2\int_{\Sigma_1}\left(\frac{\partial w}{\partial s}\right)^2 dS + \overline{d}_3\int_{\Sigma_1}\left(\frac{\partial w}{\partial n}\right)^2 dS + \overline{d}_4\int_{\Sigma_1}\left(\frac{\partial^2 w}{\partial s \partial n}\right)^2 dS + \overline{d}_5\int_{\Sigma_1}\left(\frac{\partial^2 w}{\partial s^2}\right)^2 dS. \tag{3.24}$$

A combination of (3.10), (3.20), (3.22), and (3.24) thus leads to the desired inequality, which is of the form

$$\begin{aligned}\int_R w^2 dA \leq\ & B_1\int_{\Sigma_1} w^2 dS + B_2\int_{\Sigma_1}\left(\frac{\partial w}{\partial s}\right)^2 dS + B_3\int_{\Sigma_1}\left(\frac{\partial w}{\partial n}\right)^2 dS \\ & + B_4\int_{\Sigma_1}\left(\frac{\partial^2 w}{\partial s^2}\right)^2 dS + B_5\int_{\Sigma_1}\left(\frac{\partial^2 w}{\partial s \partial n}\right)^2 dS + B_6\int_{\Sigma_2}[M(w)]^2 dS \\ & + B_7\int_{\Sigma_2}[V(w)]^2 dS + B_8\int_R [\Delta^2 w]^2 dA,\end{aligned} \tag{3.25}$$

where the B_i's are explicit constants.

In many cases (3.25) may be simplified by exploiting some of the geometric properties of Σ. For instance, whenever ρ is positive on Σ_1 some quantities may be dropped from (3.21) or used to cancel other quantities.

Pointwise bounds in this problem follow from the results of [2] and [3].

4. Problem III

This problem is the most difficult of the three since for certain geometrical configurations it does not have a unique solution. Suppose, for instance, that the domain R lies in the right half plane and Σ_1 is a portion of the y-axis. In this case the plate may rotate as a rigid body about the y-axis. The solution will be unique only up to a rigid body rotation, a term of the form $w = \alpha x$. Note, however, that if Σ_1 is not a straight line then this rigid body motion is no longer possible. Because of the inherent difficulty in treating problem 3 we shall derive in this section

only an L_2 bound for the energy integral $E(w, w)$ in terms of the data of the problem. It is possible with such a bound, using the techniques of [2, 3], to obtain pointwise bounds for any second derivative of w in terms of L_2 integrals of the data. In order to simplify the problem somewhat, we assume that w vanishes with its first two tangential derivatives at P_1 and P_2. This means merely (as pointed out before) that we put certain restrictions on the approximating functions, and it amounts to six conditions on the arbitrary parameters in the application of the Rayleigh-Ritz technique. With this restriction we decompose w as follows:

$$w = h + \tilde{w} \tag{4.1}$$

where h satisfies

$$\Delta h = 0 \text{ in } R$$
$$h = \begin{cases} w \text{ on } \Sigma_1 \\ 0 \text{ on } \Sigma_2 \end{cases}. \tag{4.2}$$

Then

$$\Delta^2 \tilde{w} = \Delta^2 w \text{ in } R$$
$$\tilde{w} = 0, M(\tilde{w}) = M(w) - (1-\sigma)\left[\frac{\partial^2 w}{\partial s^2} + \frac{1}{\rho}\frac{\partial h}{\partial n}\right] \text{ on } \Sigma_1 \tag{4.3}$$
$$M(\tilde{w}) = M(w) - \frac{1-\sigma}{\rho}\frac{\partial h}{\partial n}, V(\tilde{w}) = V(w) + (1-\sigma)\frac{\partial^3 h}{\partial s^2 \partial n} \text{ on } \Sigma_2.$$

Making use of the triangle inequality, we obtain from (4.1)

$$\{E(w, w)\}^{1/2} \leq \{E(h, h)\}^{1/2} + \{E(\tilde{w}, \tilde{w})\}^{1/2}. \tag{4.4}$$

We compute first a bound for $E(h, h)$.

Since h is harmonic it follows that

$$E(h, h) = (1-\sigma)\int_R h_{,ij} h_{,ij} dA. \tag{4.5}$$

But, as in (2.17), we observe that

$$\begin{aligned}\int_R h_{,ij} h_{,ij} dA &= \int_\Sigma \left\{ \frac{\partial h}{\partial s}\frac{\partial^2 h}{\partial s \partial n} - \frac{\partial h}{\partial n}\frac{\partial^2 h}{\partial s^2} - \frac{1}{\rho}\left[\left(\frac{\partial h}{\partial s}\right)^2 + \left(\frac{\partial h}{\partial n}\right)^2\right]\right\} dS \\ &= -2\int_\Sigma \left[\frac{\partial h}{\partial n}\frac{\partial^2 h}{\partial s^2} + \frac{1}{\rho}\left\{\left(\frac{\partial h}{\partial n}\right)^2 + \left(\frac{\partial h}{\partial s}\right)^2\right\}\right] dS.\end{aligned} \tag{4.6}$$

Applying the arithmetic-geometric mean inequality we obtain

$$\int_R h_{,ij} h_{,ij} dA \leq \int_\Sigma \alpha_9 \left(\frac{\partial^2 h}{\partial s^2}\right)^2 dS + \int_\Sigma (\alpha_9^{-1} - \rho^{-1})\left(\frac{\partial h}{\partial n}\right)^2 dS - \int_\Sigma \rho^{-1}\left(\frac{\partial h}{\partial s}\right)^2 dS. \tag{4.7}$$

If the region is convex, we could choose $\alpha_9 = \max \rho$ and drop the last two terms. If not, we may use the inequality (A.4) given in appendix A, i.e.,

$$\oint_\Sigma \left(\frac{\partial h}{\partial n}\right)^2 dS \leq b_0 \oint_\Sigma h^2 dS + b_1 \oint_\Sigma \left(\frac{\partial h}{\partial s}\right)^2 dS = b_0 \int_{\Sigma_1} u^2 dS + b_1 \int_{\Sigma_1} \left(\frac{\partial w}{\partial s}\right)^2 dS. \tag{4.8}$$

Upon inserting (4.8) and (4.7) into (4.5), and using the boundary conditions on h, we arrive at the inequality

$$E(h, h) \leq (1-\sigma)\left\{\bar{b}_0 \int_{\Sigma_1} w^2 dS + \bar{b}_1 \int_{\Sigma_1} \left(\frac{\partial w}{\partial s}\right)^2 dS + \bar{b}_2 \int_{\Sigma_1} \left(\frac{\partial^2 w}{\partial s^2}\right)^2 dS\right\} \tag{4.9}$$

where $\bar{b}_0$, $\bar{b}_1$ and $\bar{b}_2$ are explicit.

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In order to bound $E(\tilde{w}, \tilde{w})$, we decompose $\tilde{w}$ as follows:

$$\tilde{w} = B + \varphi \tag{4.10}$$

where

$$\Delta^2 B = \Delta^2 \tilde{w} \text{ in } R$$

$$M(B) = M(\tilde{w}) \text{ on } \Sigma$$

$$V(B) = \begin{cases} V(\tilde{w}) \text{ on } \Sigma_2 \\ -V(h) + \tilde{C}_1 + \tilde{C}_2 x + \tilde{C}_3 y \text{ on } \Sigma \end{cases} \tag{4.11}$$

with $\tilde{C}_1$, $\tilde{C}_2$, and $\tilde{C}_3$ chosen to insure existence of the function B. We indicate now how the $\tilde{C}_i$ are to be determined.

Let us choose the coordinate axes in such a way that $\int_{\Sigma_1} x ds = \int_{\Sigma_1} y ds = \int_{\Sigma_1} xy ds = 0$. Then, in order that (4.11) may have a solution, it is necessary that the following three equilibrium conditions be satisfied:

$$\begin{aligned} &\oint_\Sigma V(B) dS = \int_R \Delta^2 B dA \\ &\oint_\Sigma x V(B) dS - \oint_\Sigma n_x M(B) dS = \int_R x \Delta^2 B dA \\ &\oint_\Sigma y V(B) dS - \oint_\Sigma n_y M(B) dS = \int_R y \Delta^2 B dA. \end{aligned} \tag{4.12}$$

These three conditions are now used to determine $\tilde{C}_1$, $\tilde{C}_2$, and $\tilde{C}_3$. We note first that since h is harmonic

$$\begin{aligned} &\int_\Sigma V(h) dS = 0 \\ &\int_\Sigma x V(h) dS - \int_\Sigma n_x M(h) dS = 0 \\ &\int_\Sigma y V(h) dS - \int_\Sigma n_y M(h) dS = 0. \end{aligned} \tag{4.13}$$

Thus conditions (4.12) are equivalent to

$$\begin{aligned} \tilde{C}_1 S_1 &= \int_R \Delta^2 w dA - \int_{\Sigma_2} V(w) dS \\ \tilde{C}_2 I_1 &= \int_R x \Delta^2 w dA + \int_\Sigma n_x M(w) dS - \int_{\Sigma_2} x V(w) dS \\ \tilde{C}_3 I_2 &= \int_R y \Delta^2 w dA + \int_\Sigma n_y M(w) dS - \int_{\Sigma_2} y V(w) dS \end{aligned} \tag{4.14}$$

where S_1 denotes the length of Σ_1 and

$$I_1 = \int_{\Sigma_1} x^2 dS, \qquad I_2 = \int_{\Sigma_1} y^2 dS. \tag{4.15}$$

Equations (4.14) thus determine the constants $\tilde{C}_1$, $\tilde{C}_2$, and $\tilde{C}_3$ in terms of the data of problem III.

In determining the bound for $E(\tilde{w}, \tilde{w})$ of (4.4), we shall, according to (4.10), need a bound for $E(B, B)$. We first indicate how $E(B, B)$ is to be bounded in terms of the data of the problem. We present here a method which is applicable only if the domain R is simply connected. The function

B is of course determined only up to a rigid body motion, which we assume to be fixed by the conditions

$$\oint_\Sigma B dS = \oint_\Sigma \frac{\partial B}{\partial x} dS = \oint_\Sigma \frac{\partial B}{\partial y} dS = 0. \tag{4.16}$$

From the results of [3, eq (4.18)], if R is simply connected, it follows that

$$\left(\frac{2\pi}{L}\right)^2 \oint_\Sigma B^2 dS + \oint_\Sigma \left(\frac{\partial B}{\partial n}\right)^2 dS \leq \frac{2}{D\overline{p}_2(1-\sigma)} E(B, B) \tag{4.17}$$

where $\overline{p}_2$ is defined in [3, eq (4.31)].

From the identity

$$\oint_\Sigma x^i n_i B^2 dS = 2\int_R B^2 dA + 2\int_R x^i B B_{,i} dA \tag{4.18}$$

we obtain, using the arithmetic-geometric mean inequality

$$\int_R B^2 dA \leq r_M \oint_\Sigma B^2 dS + r_M^2 \int_R B_{,i} B_{,i} dA. \tag{4.19}$$

A repetition of the inequality with B^2 replaced by $B_{,i} B_{,i}$ yields

$$\int_R B^2 dA \leq r_M \oint_\Sigma B^2 dS + r_M^3 \oint_\Sigma |\operatorname{grad} B|^2 dS + r_M^4 \int_R B_{,ij} B_{,ij} dA. \tag{4.20}$$

Here r_M denotes the maximum value of $r \equiv \sqrt{x^i x^i}$ on Σ. We observed in the proof of (4.17) that

$$\begin{aligned} \oint_\Sigma B^2 dS &\leq \left(\frac{L}{2\pi}\right)^2 \oint_\Sigma \left(\frac{\partial B}{\partial s}\right)^2 dS \\ \oint_\Sigma |\operatorname{grad} B|^2 dS &\leq \frac{1}{\overline{p}_2} \int_R B_{,ij} B_{,ij} dA. \end{aligned} \tag{4.21}$$

Thus from (4.20) and (4.21) we have

$$\int_R B^2 dA \leq \nu \int_R B_{,ij} B_{,ij} dA \leq \frac{\nu}{1-\sigma} E(B, B) \tag{4.22}$$

where

$$\nu = \left\{ r_M \left(\frac{L}{2\pi}\right)^2 + r_M^3 \right\} (\overline{p}_2)^{-1} + r_M^4. \tag{4.23}$$

By Green's identity

$$\begin{aligned} E(B, B) &= \int_\Sigma \frac{\partial B}{\partial n} M(\tilde{w}) dS - \int_{\Sigma_2} B V(\tilde{w}) dS - \int_{\Sigma_1} B V(B) dS + \int_R B \Delta^2 w dA = \int_\Sigma \frac{\partial B}{\partial n} M(w) dS \\ &\quad - \int_{\Sigma_2} B V(w) dS - \int_{\Sigma_1} (\hat{C}_1 + \hat{C}_2 x + \hat{C}_3 y) B dS + \int_R B \Delta^2 w dA - \int_\Sigma \frac{\partial B}{\partial n} M(h) dS + \int_\Sigma B V(h) dS \\ &= \int_\Sigma \frac{\partial B}{\partial n} M(w) dS - \int_{\Sigma_2} B V(w) dS - \int_{\Sigma_1} (\hat{C}_1 + \hat{C}_2 x + \hat{C}_3 y) B dS + \int_R B \Delta^2 w dA - E(B, h). \end{aligned} \tag{4.24}$$

It follows then from Schwarz's inequality that

$$\begin{aligned} E(B, B) \leq &\left\{ \int_\Sigma \left(\frac{\partial B}{\partial n}\right)^2 dS + \left(\frac{2\pi}{L}\right)^2 \int_\Sigma B^2 dS \right\}^{1/2} \left\{ \int_\Sigma \left(M(w)\right)^2 dS + \left(\frac{L}{2\pi}\right)^2 \int_\Sigma \left(\overline{V}(w)\right)^2 dS \right\}^{1/2} \\ &+ \left\{ \int_R B^2 dA \int_R (\Delta^2 w)^2 dA \right\}^{1/2} + \left\{ E(B, B) \right\}^{1/2} \left\{ E(h, h) \right\}^{1/2}. \end{aligned} \tag{4.25}$$

Here $\overline{V}(w)$ is defined as

$$\overline{V}(w)=\begin{cases}V(w) \text{ on } \Sigma_2\\ \tilde{C}_1+\tilde{C}_2x+\tilde{C}_3y \text{ on } \Sigma_1.\end{cases} \tag{4.26}$$

From the definition of $\tilde{C}_1$, $\tilde{C}_2$, and $\tilde{C}_3$ we observe that $\overline{V}(w)$ is data. Using (4.17) and (4.22) we obtain then

$$\{E(B, B)\}^{1/2}\leq\left\{\frac{1}{\bar{p}_2(1-\sigma)}\left[\int_\Sigma (M(w))^2dS+\left(\frac{L}{2\pi}\right)^2\int_\Sigma(\overline{V}(w))^2dS\right]\right\}^{1/2}$$
$$+\left\{\frac{\nu}{1-\sigma}\int_R(\Delta^2w)^2dA\right\}^{1/2}+\{E(h, h)\}^{1/2}, \tag{4.27}$$

where the bound for the last term is given by (4.9). With (4.27) we now compute the bound for $E(\tilde{w}, \tilde{w})$.

From (4.10) it follows that

$$E(\tilde{w}, \tilde{w})=E(B, B)+2E(\tilde{w}, \varphi)-E(\varphi, \varphi) \tag{4.28}$$

where

$$E(\tilde{w}, \varphi)=\int_\Sigma\frac{\partial\tilde{w}}{\partial n}M(\varphi)dS-\int_\Sigma\tilde{w}V(\varphi)dS+\int_R\tilde{w}\Delta^2\varphi dA$$
$$=0 \tag{4.29}$$

from (4.10) and (4.11). Thus

$$E(\tilde{w}, \tilde{w})\leq E(B, B). \tag{4.30}$$

Combining (4.4), (4.9), and (4.27) we now obtain (after an application of the arithmetic-geometric mean inequality) an inequality of the form

$$E(w, w)\leq D_1\int_{\Sigma_1}w^2dS+D_2\int_{\Sigma_1}\left(\frac{\partial w}{\partial s}\right)^2dS+D_3\int_{\Sigma_1}\left(\frac{\partial^2w}{\partial s^2}\right)^2dS$$
$$+D_4\oint_\Sigma(M(w))^2dS+D_5\oint_\Sigma(\overline{V}(w))^2dS+D_6\int_R(\Delta^2w)^2dA, \tag{4.31}$$

which is the desired result.

5. Appendix A

In treating the problems discussed in sections 2, 3, and 4 we required bounds for certain integrals of derivatives of biharmonic functions in terms of their Dirichlet data. We derive the necessary inequalities here, the most important of which is (A.25).

Consider the following Dirichlet problem for R:

$$\Delta^2B=0 \text{ in } R$$
$$B, \frac{\partial B}{\partial n} \text{ given on } \Sigma. \tag{A.1}$$

Let us compute a bound for $\int_R(\Delta B)^2dA$ in terms of the data of B. We assume that the data are such that $\partial^2B/\partial s^2$ and $\partial^2B/\partial s\partial n$ are square integrable on Σ.

Let h be the harmonic function which is equal to B on the boundary. Then from Green's identity we have

$$\int_R (\Delta B)^2 dA = \int_R [\Delta(B-h)]^2 dA = \int_\Sigma \frac{\partial}{\partial n}(B-h)\Delta B dS. \tag{A.2}$$

Using the arithmetic-geometric mean inequality we obtain, for any positive α_0, the inequality

$$\int_R (\Delta B)^2 dA \leq \frac{\alpha_0}{2}\oint_\Sigma (\Delta B)^2 dS + \frac{1}{2\alpha_0}\oint_\Sigma \left[\frac{\partial}{\partial n}(B-h)\right]^2 dS$$

$$\leq \frac{\alpha_0}{2}\oint_\Sigma (\Delta B)^2 dS + \frac{1}{\alpha_0}\left[\oint_\Sigma \left(\frac{\partial B}{\partial n}\right)^2 dS + \oint_\Sigma \left(\frac{\partial h}{\partial n}\right)^2 dS\right]. \tag{A.3}$$

In [10, 11] Payne and Weinberger have obtained the following inequality for the last integral in (A.3):

$$\oint_\Sigma \left(\frac{\partial h}{\partial n}\right)^2 dS \leq b_0\oint_\Sigma h^2 dS + b_1\oint_\Sigma \left(\frac{\partial h}{\partial s}\right)^2 ds = b_0\oint_\Sigma B^2 ds + b_1\oint_\Sigma \left(\frac{\partial B}{\partial s}\right)^2 dS. \tag{A.4}$$

The constants b_0 and b_1 are explicitly determined. We seek now a bound for the first integral on the right of (A.3).

For an arbitrary sufficiently smooth vector field g^i and an arbitrary harmonic function H we consider the following identity:

$$0 = \int_R g^k (B-H)_{,k}\Delta^2 B dA = \oint_\Sigma g^k (B-H)_{,k}\frac{\partial}{\partial n}(\Delta B) dS$$

$$- \int_R g^k{}_{,l}^{l}(B-H)_{,k}\Delta B_{,l} ds - \int_R g^k (B-H)_{,kl}\Delta B_{,l} dA. \tag{A.5}$$

Integrating by parts further we obtain

$$\int_\Sigma g^i \Delta B\left[n_k \frac{\partial}{\partial x^i} - n_i \frac{\partial}{\partial x^k}\right](B-H)_{,k} dS + \frac{1}{2}\oint_\Sigma g^k n_k (\Delta B)^2 dS + \int_\Sigma g^k (B-H)_{,k}\frac{\partial(\Delta B)}{\partial n} dS$$

$$+ \int_\Sigma g,{}_k^i n_k (B-H)_{,i}\Delta B dS = \int_R \left\{\Delta g^i (B-H)_{,i} + 2g,{}_k^i (B-H)_{,ik} - \frac{1}{2} g,{}_i^i \Delta B\right\}\Delta B dA. \tag{A.6}$$

Suppose now that a vector function g^i with bounded second derivatives has been found such that

$$g^i = n_i \tag{A.7}$$

on Σ. If $\Sigma \epsilon C^2$ one possible vector field g^i is given in appendix B. With g^i so chosen we take H to satisfy

$$\Delta H = 0 \text{ in } R$$

$$\frac{\partial H}{\partial n} = \frac{\partial B}{\partial n} - S^{-1}\oint_\Sigma \frac{\partial B}{\partial n} dS \tag{A.8}$$

where S denotes the length of Σ. From (A.7) and (A.8) it follows that the third boundary integral on the left of (A.6) vanishes (since B is biharmonic).

Since $g^i = n_i$ on Σ, the first term in (A.6) becomes

$$\oint_\Sigma n_i \Delta B\left(n_k \frac{\partial}{\partial x^i} - n_i \frac{\partial}{\partial x^k}\right)(B-H)_{,k} dS = \oint_\Sigma \Delta B\left[-\frac{\partial^2}{\partial s^2} - \frac{1}{\rho}\frac{\partial}{\partial n}\right](B-H) dS \tag{A.9}$$

while the fourth term on the left may be written as

$$\oint_\Sigma n_{i,\,k} n_k (B-H)_{,i}\Delta B dS = \oint_\Sigma \frac{\partial g^i}{\partial n}(B-H)_{,i}\Delta B dS. \tag{A.10}$$

Inserting (A.9) and (A.10) into (A.6), and making use of (A.7) and (A.8), we obtain

$$\frac{1}{2}\oint_\Sigma (\Delta B)^2 dS = \oint_\Sigma \Delta B \frac{\partial^2}{\partial s^2}(B-H)dS + S^{-1}\oint_\Sigma (\Delta B)\rho^{-1}dS \oint_\Sigma \frac{\partial B}{\partial n}dS - \oint_\Sigma \frac{\partial g^i}{\partial n}\left\{\frac{\partial}{\partial s}(B-H)\frac{\partial x^i}{\partial s}\right.$$
$$\left. + n_i S^{-1}\oint_\Sigma \frac{\partial B}{\partial n}dS\right\}\Delta B dS + \int_R \Delta B\left\{\Delta g^i (B-H)_{,i} + 2g^i_{;k}(B-H)_{,ik} - \frac{1}{2}g,^i_i \Delta B\right\}dA. \qquad (A.11)$$

We assume now that g^i and its second derivatives are uniformly bounded in R. Then by making use of the arithmetic-geometric mean inequality we obtain for arbitrary positive constants α_i

$$(1-\alpha_1-\alpha_2-\alpha_3-\alpha_4)\oint_\Sigma (\Delta B)^2 dS \leq \frac{1}{\alpha_1}\oint_\Sigma \left[\frac{\partial^2}{\partial s^2}(B-H)\right]^2 dS + \frac{(p^{-2})_{\max}}{\alpha_2}\oint_\Sigma \left(\frac{\partial B}{\partial n}\right)^2 dS$$
$$+ \frac{C_1}{\alpha_3}\oint_\Sigma \left[\frac{\partial}{\partial s}(B-H)\right]^2 dS + \frac{C_2}{\alpha_4}\oint_\Sigma \left(\frac{\partial B}{\partial n}\right)^2 dS + (\alpha_5+\alpha_6+C_3)\int_R (\Delta B)^2 dA$$
$$+ \frac{C_4}{\alpha_5}\int_R (B-H),_i (B-H),_i dA + \frac{C_5}{\alpha_6}\int_R (B-H),_{ij}(B-H),_{ij}dA. \qquad (A.12)$$

In (A.12) we have used the abbreviations

$$C_1 = \max_\Sigma \left[\frac{\partial g^i}{\partial n}\frac{\partial x^i}{\partial s}\right]^2$$
$$C_2 = \max_\Sigma \left[\frac{\partial g^i}{\partial n}n_i\right]^2$$
$$C_3 = \max_R (g,^i_i)^2$$
$$C_4 = \max_R [\Delta g^i \Delta g^i]^2$$
$$C_5 = 4\max_R |g, ^i_k g, ^k_i|. \qquad (A.13)$$

We need now bounds for $\int_R (B-H),_i (B-H),_i dA$, $\int_R (B-H),_{ij}(B-H),_{ij}dA$ and L_2 integrals over Σ of the first and second tangential derivatives of H, in terms of the data of B. We consider first $\int_R (B-H),_{ij}(B-H),_{ij}dA$. An integration by parts gives (see 2.17)

$$\int_R (B-H),_{ij}(B-H),_{ij}dA = 2\oint_\Sigma \frac{\partial^2 (B-H)}{\partial s \partial n}\frac{\partial (B-H)}{\partial s}dS$$
$$+ \int_R (\Delta B)^2 dA - \oint_\Sigma \frac{1}{\rho}[(B-H),_i (B-H),_i]dS. \qquad (A.14)$$

The first term on the right vanishes in view of (A.8). Thus

$$\int_R (B-H),_{ij}(B-H),_{ij}dA = \int_R (\Delta B)^2 dA - S^{-2}\oint_\Sigma \frac{1}{\rho}dS\left(\oint_\Sigma \frac{\partial B}{\partial n}ds\right)^2 - \oint_\Sigma \frac{1}{\rho}\left[\frac{\partial}{\partial s}(B-H)\right]^2 dS$$
$$\leq \int_R (\Delta B)^2 dA + C_6\oint_\Sigma \left(\frac{\partial B}{\partial n}\right)^2 dS + C_6\oint_\Sigma \left[\frac{\partial}{\partial s}(B-H)\right]^2 dS \qquad (A.15)$$

where C_6 is given by

$$C_6 = \begin{cases} 0 \text{ if } \rho \geq 0 \text{ on } \Sigma \\ \left(-\frac{1}{\rho}\right)_{\max} \text{ if } \rho \text{ changes sign on } \Sigma. \end{cases} \qquad (A.16)$$

In a recent paper [6] the authors obtained the following inequality:

$$\int_R (B-H),_i(B-H),_i dA \leq C_7 \oint_\Sigma \left[\frac{\partial}{\partial n}(B-H)\right]^2 dS + C_8 \int_R (\Delta B)^2 dA$$

$$\leq C_7 \oint_\Sigma \left(\frac{\partial B}{\partial n}\right)^2 dS + C_8 \int_R (\Delta B)^2 dA \qquad \text{(A.17)}$$

with explicit constants C_7 and C_8. The second inequality results from insertion of the boundary condition in (A.8) followed by an application of Schwarz's inequality. The insertion of (A.15) and (A.17) into (A.12) then gives

$$(1-\alpha_1-\alpha_2-\alpha_3-\alpha_4)\oint_\Sigma (\Delta B)^2 dS \leq \frac{1}{\alpha_1}\oint_\Sigma \left[\frac{\partial^2}{\partial s^2}(B-H)\right]^2 dS$$

$$+\left[\frac{(\rho^{-2})_{\max}}{\alpha_2}+\frac{C_2}{\alpha_4}+\frac{C_5C_6}{\alpha_6}+\frac{C_4C_7}{\alpha_5}\right]\oint_\Sigma \left(\frac{\partial B}{\partial n}\right)^2 dS+\left[\frac{C_1}{\alpha_3}+\frac{C_5C_6}{\alpha_6}\right]\oint_\Sigma\left[\frac{\partial}{\partial s}(B-H)\right]^2 dS$$

$$+\left[\alpha_5+\alpha_6+C_3+\frac{C_5}{\alpha_6}+\frac{C_4C_8}{\alpha_5}\right]\int_R (\Delta B)^2 dA. \qquad \text{(A.18)}$$

We now choose the constants α_i in such a way that

$$\gamma \equiv 1-\alpha_1-\alpha_2-\alpha_3-\alpha_4 > 0. \qquad \text{(A.19)}$$

For simplicity we write (A.18) as

$$\gamma_1 \oint_\Sigma (\Delta B)^2 dS \leq \gamma_2 \oint_\Sigma \left[\frac{\partial^2}{\partial s^2}(B-H)\right]^2 dS + \gamma_3 \oint_\Sigma \left(\frac{\partial B}{\partial n}\right)^2 dS$$

$$+\gamma_4 \oint_\Sigma \left[\frac{\partial}{\partial s}(B-H)\right]^2 dS + \gamma_5 \int_R (\Delta B)^2 dA. \qquad \text{(A.20)}$$

We now insert (A.4) and (A.20) back into (A.3) to obtain

$$\left(\gamma_1-\frac{\alpha_0\gamma_5}{2}\right)\int_R (\Delta B)^2 dA \leq \gamma_2 \oint_\Sigma \left[\frac{\partial^2}{\partial s^2}(B-H)\right]^2 dS + \left(\gamma_3+\frac{\gamma_1}{\alpha_0}\right)\oint_\Sigma \left(\frac{\partial B}{\partial n}\right)^2 dS$$

$$+\gamma_4 \oint_\Sigma \left[\frac{\partial}{\partial s}(B-H)\right]^2 dS + \frac{\gamma_1 b_0}{\alpha_0}\oint_\Sigma B^2 dS + \frac{\gamma_1 b_1}{\alpha_6}\oint_\Sigma \left(\frac{\partial B}{\partial s}\right)^2 dS. \qquad \text{(A.21)}$$

The constant α_0 may be taken as γ_1/γ_5 which yields the inequality:

$$\frac{\gamma_1}{2}\int_R (\Delta B)^2 dA \leq \gamma_2 \oint_\Sigma \left[\frac{\partial^2}{\partial s^2}(B-H)\right]^2 dS + (\gamma_3+\gamma_5)\oint_\Sigma \left(\frac{\partial B}{\partial n}\right)^2 dS$$

$$+\gamma_4 \oint_\Sigma \left[\frac{\partial}{\partial s}(B-H)\right]^2 dS + b_0\gamma_5 \oint_\Sigma B^2 dS + b_1\gamma_5 \oint_\Sigma \left(\frac{\partial B}{\partial s}\right)^2 dS. \qquad \text{(A.22)}$$

We need now to bound the tangential derivatives of H in terms of its normal derivatives. From the results of [10, 11] we have

$$\oint_\Sigma \left(\frac{\partial H}{\partial s}\right)^2 dS \leq \tilde{C}_9 \oint_\Sigma \left(\frac{\partial H}{\partial n}\right)^2 dS + \bar{C}_9 \int_R H,_i H,_i dA \leq C_9 \int_\Sigma \left(\frac{\partial B}{\partial n}\right)^2 dS \qquad \text{(A.23)}$$

with explicit constants $\tilde{C}_9$, $\bar{C}_9$, and C_9. An inequality of type (A.17) has been used to obtain (A.23).

We require finally a bound for $\oint_\Sigma \frac{\partial^2 H}{\partial s^2} dS$.

Since H is harmonic we obtain, on applying an inequality of Bramble and Hubbard [1],

$$\oint_\Sigma \left[\frac{\partial^2 H}{\partial s^2}\right]^2 dS \leq \tilde{C}_{10}\oint_\Sigma \left[\frac{\partial^2 H}{\partial s \partial n}\right]^2 dS + \tilde{C}_{11}\oint_\Sigma \left(\frac{\partial H}{\partial n}\right)^2 dS \leq C_{10}\oint_\Sigma \left[\frac{\partial^2 B}{\partial s \partial n}\right]^2 dS + C_{11}\oint_\Sigma \left(\frac{\partial B}{\partial n}\right)^2 dS \tag{A.24}$$

with explicit constants C_{10} and C_{11} (see Appendix C). A use of the arithmetic-geometric mean inequality in the first and third terms on the right of (A.22), together with (A.23) and (A.24), yields the inequality

$$\begin{aligned}\frac{\gamma_1}{2}\int_R (\Delta B)^2 dA \leq\ & 2\gamma_2 \oint_\Sigma \left(\frac{\partial^2 B}{\partial s^2}\right)^2 dS + 2\gamma_2 C_{10}\oint_\Sigma \left(\frac{\partial^2 B}{\partial s \partial n}\right)^2 dS + (2\gamma_4 + b, \gamma_5)\oint_\Sigma \left(\frac{\partial B}{\partial s}\right)^2 dS \\ & + (\gamma_3 + \gamma_5 + 2\gamma_2 C_{11} + 2\gamma_4 C_9)\oint_\Sigma \left(\frac{\partial B}{\partial n}\right)^2 dS + b_0\gamma_5 \oint_\Sigma B^2 dS \qquad \text{(A.25)} \\ \leq\ & d_1 \oint_\Sigma B^2 dS + d_2 \oint_\Sigma \left(\frac{\partial B}{\partial s}\right)^2 dS + d_3 \oint_\Sigma \left(\frac{\partial B}{\partial n}\right)^2 dS + d_4 \oint_\Sigma \left[\frac{\partial B_{,i}}{\partial s}\frac{\partial B_{,i}}{\partial s}\right] dS\end{aligned}$$

which was to be derived.

It might be noted that the left hand side is unaffected by considering $\tilde{B}$ instead of B, where

$$\tilde{B} = B + a_0 + a_1 x + a_2 y. \tag{A.26}$$

In (A.26) the constants a_i are determined by the conditions that

$$\oint_\Sigma \tilde{B} dS = \oint_\Sigma \tilde{B}_{,x} dS = \oint_\Sigma \tilde{B}_{,y} dS = 0 \tag{A.27}$$

Then, as indicated in [3], it is possible, if R is simply connected, to replace (A.25) by

$$\int_R (\Delta B)^2 dA \leq d_5 \oint_\Sigma \frac{\partial \tilde{B}_{,i}}{\partial s}\frac{\partial \tilde{B}_{,i}}{\partial s} dS = d_5 \oint_\Sigma \frac{\partial B_{,i}}{\partial s}\frac{\partial B_{,i}}{\partial s} dS. \tag{A.28}$$

6. Appendix B. The Vector Field g^i

We make the assumption that $\Sigma \epsilon C^3$ and consider the family of parallel surfaces to Σ in R given by $N(x) = \text{constant}$, with Σ defined by $N(x) = 0$. The parallel surfaces need only be defined in the neighborhood of Σ and we shall be interested in the subdomain of R given by $0 \leq N(x) \leq K^{-1}$ where K is a bound on the curvature of Σ. The outward normal vector n_i on the parallel surface is given by

$$n_i = \frac{-N_{,i}}{\{N_{,j}N_{,j}\}^{1/2}}.$$

It is clear then that if we define g^i as

$$g^i = \begin{cases} (1 - KN)^2 n_i, & 0 \leq N \leq K^{-1} \\ 0 \text{ otherwise} \end{cases}$$

then $g^i = n_i$ on Σ and $g^i \epsilon C^2$ in R.

7. Appendix C

The inequality (A.24) follows from eq (2.6) of [1] by making use of [1, eq (2.2)] to prescribe the arbitrary vectors g^i and symmetric tensors A^{ij}. We sketch here briefly how the argument goes.

For sufficiently smooth g^i and A^{ij} we have the following identity for any harmonic function H:

$$\int_R [2(g^m A^{ij})_{,l} H_{,jm} - (g^m A^{ij})_{,m} H_{,jl}] H_{,il} dA = \oint_\Sigma g^m A^{ij} H_{,il}(2H_{,jm} n_l - H_{,jl} n_m) dS. \tag{C.1}$$

Suppose g^i and A^{ij} to be so chosen that on Σ

$$g^i = n_i$$

$$A^{ij} = \delta_{ij} - n_i n_j \tag{C.2}$$

If Σ is sufficiently smooth, we may choose g^i as in appendix B and

$$A^{ij} = \delta_{ij} - g^i g^j.$$

We now decompose the derivatives on the boundary into normal and tangential derivatives to obtain

$$\oint_\Sigma [n_m \{\delta_{ij} - n_i n_j\} H,_{ii} (2H,_{jm} n_l - H,_{jl} n_m)] dS$$

$$= \oint_\Sigma \left\{ \left(\frac{\partial^2 H}{\partial s \partial n}\right)^2 - \left(\frac{\partial^2 H}{\partial s^2}\right)^2 - \frac{2}{\rho}\frac{\partial H}{\partial s}\frac{\partial^2 H}{\partial s \partial n} + \frac{1}{\rho^2}\left(\frac{\partial H}{\partial s}\right)^2 \right.$$

$$\left. - \left(\frac{\partial n_1}{\partial s}\frac{\partial H}{\partial s} - \frac{\partial n_2}{\partial n}\frac{\partial H}{\partial n}\right)^2 - \left(\frac{\partial n_2}{\partial s}\frac{\partial H}{\partial s} + \frac{\partial n_1}{\partial n}\frac{\partial H}{\partial n}\right)^2 \right\} dS. \tag{C.3}$$

Inserting (C.3) back into (C.1), we have upon rearranging and dropping the last two terms

$$\oint_\Sigma \left(\frac{\partial^2 H}{\partial s^2}\right)^2 dS \leq \oint_\Sigma \left\{ \left(\frac{\partial^2 H}{\partial s \partial n}\right)^2 - \frac{2}{\rho}\frac{\partial H}{\partial s}\frac{\partial^2 H}{\partial s \partial n} + \frac{1}{\rho^2}\left(\frac{\partial H}{\partial s}\right)^2 \right\} dS + \bar{k} \int_R H,_{ij} H,_{ij} dA, \tag{C.4}$$

where $\bar{k}$ is easily determined once g^i and A^{ij} are given. From (A.8) it follows that

$$\oint_\Sigma \left(\frac{\partial^2 H}{\partial s^2}\right)^2 dS \leq \oint_\Sigma \left\{ \left(\frac{\partial^2 B}{\partial s \partial n}\right)^2 - \frac{2}{\rho}\frac{\partial H}{\partial s}\frac{\partial^2 B}{\partial s \partial n} + \frac{1}{\rho^2}\left(\frac{\partial H}{\partial s}\right)^2 \right\} dS + \bar{k} \int_R H,_{ij} H,_{ij} dA. \tag{C.5}$$

But from (4.16)

$$\int_R H,_{ij} H,_{ij} dA = 2 \oint_\Sigma \frac{\partial^2 B}{\partial s \partial n}\frac{\partial H}{\partial s} dS - \oint_\Sigma \frac{1}{\rho}\left[\left(\frac{\partial H}{\partial n}\right)^2 + \left(\frac{\partial H}{\partial s}\right)^2\right] dS.$$

If we now use the arithmetic-geometric mean inequality in the second term on the right of (C.5) and the second term on the right of (C.6), we obtain an inequality of the form

$$\oint_\Sigma \left(\frac{\partial^2 H}{\partial s^2}\right)^2 dS \leq \bar{K}_1 \oint_\Sigma \left(\frac{\partial^2 B}{\partial s \partial n}\right)^2 dS + \bar{K}_2 \oint_\Sigma \left(\frac{\partial H}{\partial s}\right)^2 dS + \bar{K}_3 \oint_\Sigma \left(\frac{\partial H}{\partial n}\right)^2 dS. \tag{C.7}$$

We now use (A.23) and apply Schwarz's inequality to the last term on the right after insertion of the boundary condition. This leads to (A.24).

8. References

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Selected Abstracts

The Pythagorean theorem in certain symmetry classes of tensors, M. Marcus and H. Minc, *Trans. Am. Math. Soc.* **104,** *No. 3, 510–515 (Sept. 1962).*
The authors prove that if $A=(a_{ij})$ is a symmetric positive semi-definite $n\times n$ matrix, then the permanent of A satisfies

$$\text{per } (A) \geq (a_{11} \ldots a_{nn})\frac{n!}{n^{2n}}.$$

Error bounds for first approximations in turning-point problems, F. W. J. Olver, *J. Soc. Indust. Appl. Math.* **11,** *No. 3, 748–772 (Sept. 1963).*
Error bounds are derived and examined for approximate solutions in terms of Airy functions of the differential equations

$$\frac{d^2w}{dx^2}=x+f(x)w\ ,\ \frac{d^2w}{dx^2}=p(x)+q(x)w.$$

The variable x ranges over a real interval which may be finite or infinite. The functions $f(x)$, $p(x)$ and $q(x)$ are free from singularities and $p(x)$ has just one zero in the interval.
An application is made to Hermite polynomials of large degree, and numerical examples are included.

Foundations of the Callaway theory of thermal conductivity, R. E. Nettleton, *Phys. Rev.* **132,** *No. 5, 2032–2038 (Dec. 1, 1963).*
The equation derived by Peierls for the occupation numbers N_q of phonon states with wave vector $\mathbf{q}$ and appropriate polarization is linearized to give a master equation for the deviations $n_q \equiv N_q - N_q^0$ from equilibrium. This master equation, with transition probabilities calculated from first-order time-dependent perturbation theory, is then compared term-by-term with the linearized form of an approximate master equation proposed by Callaway. The latter assumes that the effects of normal and umklapp processes can be represented by two relaxation frequencies, τ_N^{-1} and τ_r^{-1} respectively, which are both proportional to q^2. This assumption is contradicted by the present analysis, which shows that the condition that the solution of the Callaway equation should also satisfy the Peierls equation in the case of steady-state heat flow leads to τ_N^{-1} proportional to q in first approximation. Furthermore, the Callaway equation contains non-zero transition probabilities for a great many phonon processes which are absent from the Peierls equation. Thus, while Callaway's equation, with a modified τ_N, can indeed give a good approximation to the phonon occupation numbers in the steady state, it does not describe the approach to that state from an arbitrary distribution.

Numerical solutions of the convolution-hypernetted chain integral equation for the pair correlation function of a fluid. I. The Lennard-Jones (12,6) potential, M. Klein and M. S. Green, *J. Chem. Phys.* **39,** *No. 6, 1367–1387 (Sept. 15, 1963).*
The integral equation of the convolution-hypernetted chain approximation is solved numerically at five temperatures for the Lennard-Jones (12, 6) potential. Due to an inconsistency in the approximation there are two equations of state associated with it. These are compared with each other, with other theories and with the experimental argon equation of state. At high temperatures, the two equations of state agree with both experiment and more a priori theory at low densities and bracket them at higher densities, their mean giving a good representation of both. The integral equation is found to be singular at certain temperature-density points, their locus forming a dome shaped curve in the $T-\rho$ plane. These points are shown to correspond to the limits of metastability in the van der Waals' gas, the highest temperature point being the critical point. The non-singular points below the critical point correspond either to a homogeneous gas (low densities) or liquid (high densities) phase. The former are in good agreement with the three term virial series, as expected, and the latter are a considerable improvement over it. The equation of state is a considerable improvement over that obtained from the Born-Green equation by Kirkwood, Lewinson, and Alder, the latter predicting negative pressures over most of the liquid range while these results show only positive pressures. The comparisons with experiment are obscured by the inadequacy of the (12, 6) potential function. The use of temperature "compensated" potential parameters (as determined by the second virial coefficient) indicates that the use of a more reasonable potential function in the theory will give a relatively good representation of the equation of state of a fluid over a wide range of temperatures and densities.

Numerical solutions of the convolution-hypernetted chain integral equation for the pair correlation function of a fluid. II. The hard-sphere potential, M. Klein, *J. Chem. Phys.* **39,** *No. 6, 1388–1397 (Sept. 15, 1963).*
The nonlinear integral equation of the hypernetted chain (HNC) approximation is solved for a fluid of hard spheres. An inconsistency introduced by the approximation results in a disagreement between two forms of the equation of state. The resulting pair of equations of state are compared with each other, with those obtained for the Born-Green and (approximate) Kirkwood equations by Kirkwood, Maun, and Alder, and with a reference isotherm made up of a combination of the five-term virial series at low and intermediate densities and the gas-like isotherm of molecular dynamics at high densities. The HNC equations of state are found to be a considerable improvement over those of the Born-Green and Kirkwood theories. At intermediate and high densities, the two HNC isotherms bracket the reference isotherm, their mean being a very good approximation to it. The HNC equation exhibits a tendency toward a gas-solid phase transition at extremely high densities. A transition is not actually predicted, however.

The three-body scattering operator in nonequilibrium statistical mechanics, J. Weinstock, *Phys. Rev.* **132,** *No. 1, 470–482 (Oct. 1, 1963).*
A method is presented for calculating the time-dependent irreducible clusters, $\beta_s(t)$ which appear in the kernel of the equation of evolution derived in the preceding article. The clusters $\beta_1(t)$ and $\beta_2(t)$—which correspond to binary and ternary collisions, respectively—are calculated in detail. They are each found to divide into the two following parts: (1) a "completed" collision part which corresponds to collisions which are eventually completed (scattering processes) and (2) an "incompleted" part which corresponds to those collisions not completed by time t. The incompleted collision parts contribute to the "memory" of the equation of evolution and are shown to be relatively small when t is large. The completed collision parts, which play a central role in the theory of transport coefficients, are time-independent scattering operators in momentum space and do not contribute to the memory. By means of the "binary-collision expansion" a systematic method is presented for the calculation of the three-body scattering operator $[\lim_{t\to\infty} t^{-1}\beta_2(t)]$ which is directly applicable to interaction forces with infinite repulsions. An approximate formula is then derived for this scattering operator in a form which can be readily used to calculate the density correction to transport coefficients which arise from ternary collisions.

Cluster formulation of the exact equation for the evolution of a classical many-body system, J. Weinstock, *Phys. Rev.* **132,** *No. 1, 454–469 (Oct. 1, 1963).*
An exact non-Markoffian equation is derived for the evolution of an infinite homogeneous system. This equation—which may be viewed as a time-dependent analog of the equilibrium virial expansion—may be readily applied when the forces between particles include infinite repulsions. The derivation of this equation from Liouville's equation is analogous to Mayer's derivation of the virial expansion from the partition function. In this way the formal development of nonequilibrium statistical mechanics is placed on a similar footing

to that of equilibrium statistical mechanics, and a many-body problem is reduced to understanding the dynamics of isolated groups of particles. Fourier expansions and expansions in powers of the interaction potential are avoided by dealing with s-body Green functions (propagators) which are always convergent functions of the interaction potential. These functions correspond to multiplet collisions in ordinary configuration space between s isolated particles and are time-dependent analogs of the irreducible clusters well known in equilibrium statistical mechanics. The kernel (memory) of the equation of evolution consists of a linear sum of the time-dependent irreducible clusters. The non-Markoffian behavior of the equation of evolution is, thus, directly given by the time dependence of these clusters, and is explicitly related to incompleted collisions. The equation of evolution is solved in the asymptotic limit of long times. In this limit it is found (because the kernel rapidly vanishes) that the equation reduces to a Markoffian master equation involving a scattering operator for both completed and incompleted collisions in configuration space.

Two assumptions in the theory of attractive forces between long saturated chains, R. Zwanzig, *J. Chem. Phys.* **39,** *No. 9, 2251–2258 (Nov. 1, 1963).*

A recent calculation by Salem of the attractive forces between two long parallel hydrocarbon chains was based on two assumptions: (1) pairwise additivity of dispersion forces between individual groups on each chain, and (2) isotropy of the polarizability of a group on the chain. The validity of these assumptions is examined here by means of a Drude model calculation of the interaction between two parallel linear lattices of dispersion oscillators. We describe an exact calculation on this model, and also a calculation based on applying Salem's assumptions to the same model. We conclude that the assumptions are not valid for the Drude model, because of strong induced-dipole interactions between neighbors along the lattice. Salem's procedure leads to essentially correct numerical results, however, due to use of an experimentally determined bond polarizability at a crucial stage of the calculation. This allows for the effects of interactions between neighbors. Our results are applicable to real hydrocarbon chains only to the extent that the Drude model is valid.

Zeros of polynomials and fractional order differences of their coefficients, G. T. Cargo and O. Shisha, *J. Math. Anal. Appl.* **7,** *No. 2, 176–182 (Oct. 1963).*

Let $E(z) \equiv \sum_{k=0}^{n} c_k z^k (\not\equiv 0,\ n \geq 1)$ be a polynomial, and let $0 < \alpha \leq 1$.

Assume that $c_k \geq 0 (k = 0, 1, \ldots, n)$ and that $\nabla^\alpha c_k \leq 0 (k = 1, 2, \ldots, n)$.

Then no zero of $E(z)$ lies in $|z| < 1$. Here $\nabla^\alpha c_k \equiv \sum_{m=0} (-1)^m \binom{\alpha}{m} c_{k-m}$, i.e., $\nabla^\alpha c_k$ is a (backwards) difference of order α. The theorem generalizes a well-known result of Eneström, and can be further generalized to polynomials and power series with complex coefficients.

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Tritium-labeled compounds. XI. Mechanism for the oxidation of aldehydes and aldoses-*l*-*t* with sodium chlorite. H. S. Isbell and L. T. Sniegoski.

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